A PVS Library for Measure and Integration

David Lester

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Contents

Part I
Local Extras

1 partitions

```
partitions[T: TYPE]: THEORY
 BEGIN

  a, a1, a2: VAR set[T]

  A: VAR setofsets[T]

  partition?(A, a): bool =
        ⋃A = a ∧
         (∀ (x, y: (A)): x ≠ y ⇒ disjoint?(x, y))

  finite_partition?(A, a): bool =
        partition?(A, a) ∧ is_finite(A)

  partition: TYPE+ = (λ A: partition?(A, fullset[T])) CONTAINING singleton[set[T]]
                                                                        (fullset
                                                                          [T])

  finite_partition: TYPE+ = (λ A: finite_partition?(A, fullset[T])) CONTAINING singleton
                                                                                      [set
                                                                                        [T]]
                                                                                      (fullset
                                                                                        [T])

  p1, p2: VAR finite_partition

  IMPORTING finite_sets@finite_cross

  join(p1, p2): finite_partition =
        {a | ∃ (a1: (p1), a2: (p2)): a = (a1 ∩ a2)}

 END partitions
```

2 pointwise_convergence

pointwise_convergence[T: TYPE]: THEORY
BEGIN

IMPORTING metric_space@convergence_aux

u, v: VAR sequence[[$T \to \mathbb{R}$]]

f, f_0, f_1: VAR [$T \to \mathbb{R}$]

g: VAR [$T \to \mathbb{R}_{\geq 0}$]

x: VAR T

c: VAR $\mathbb{R}$

n, m: VAR $\mathbb{N}$

P: VAR pred[sequence[$\mathbb{R}$]]

zero_seq(n)(x): $\mathbb{R}$ = 0

pointwise?(P)(u): bool = $\forall$ x: $P(\lambda$ n: $u(n)(x))$

pointwise_bounded_above?(u): bool =
pointwise?(bounded_above?)(u)

pointwise_bounded_below?(u): bool =
pointwise?(bounded_below?)(u)

pointwise_bounded?(u): bool = pointwise?(bounded_seq?)(u)

pointwise_bounded_def: LEMMA
pointwise_bounded?(u) $\Leftrightarrow$
(pointwise_bounded_above?(u) $\wedge$ pointwise_bounded_below?(u))

pointwise_bounded_above: TYPE+ = (pointwise_bounded_above?) CONTAINING zero_seq

pointwise_bounded_below: TYPE+ = (pointwise_bounded_below?) CONTAINING zero_seq

pointwise_bounded: TYPE+ = (pointwise_bounded?) CONTAINING zero_seq

pointwise_bounded_is_bounded_above: JUDGEMENT pointwise_bounded SUBTYPE_OF
pointwise_bounded_above

pointwise_bounded_is_bounded_below: JUDGEMENT pointwise_bounded SUBTYPE_OF
pointwise_bounded_below

$u \longrightarrow f$: bool = $\forall$ x: λ n : $u(n)(x) \longrightarrow f(x)$

pointwise_convergent?(u): bool = $\exists\ f:\ u \longrightarrow f$

pointwise_convergent: TYPE+ = (pointwise_convergent?) CONTAINING zero_seq

IMPORTING reals@real_fun_ops_aux[T]

$u + v$: sequence[[$T \rightarrow \mathbb{R}$]] =
$\lambda\ n:\ u(n) + v(n)$;

$c \times u$: sequence[[$T \rightarrow \mathbb{R}$]] = $\lambda\ n:\ c \times u(n)$;

$-u$: sequence[[$T \rightarrow \mathbb{R}$]] = $\lambda\ n:\ -(u(n))$;

$u - v$: sequence[[$T \rightarrow \mathbb{R}$]] =
$\lambda\ n:\ u(n) - v(n)$;

u^{+}: sequence[[$T \rightarrow \mathbb{R}_{\geq 0}$]] = $\lambda\ n:\ u(n)^{+}$;

u^{-}: sequence[[$T \rightarrow \mathbb{R}_{\geq 0}$]] = $\lambda\ n:\ u(n)^{-}$;

pointwise_convergence_sum: LEMMA
$u \longrightarrow f_0 \ \wedge\ v \longrightarrow f_1 \ \Rightarrow\ u + v \longrightarrow f_0 + f_1$

pointwise_convergence_scal: LEMMA
$u \longrightarrow f \ \Rightarrow\ c \times u \longrightarrow c \times f$

pointwise_convergence_opp: LEMMA $u \longrightarrow f \ \Rightarrow\ -u \longrightarrow -f$

pointwise_convergence_diff: LEMMA
$u \longrightarrow f_0 \ \wedge\ v \longrightarrow f_1 \ \Rightarrow\ u - v \longrightarrow f_0 - f_1$

w, w_0, w_1: VAR pointwise_convergent

pointwise_convergent_sum: JUDGEMENT $+(w_0,\ w_1)$ HAS_TYPE
pointwise_convergent

pointwise_convergent_scal: JUDGEMENT $\times(c,\ w)$ HAS_TYPE
pointwise_convergent

pointwise_convergent_opp: JUDGEMENT $-(w)$ HAS_TYPE
pointwise_convergent

pointwise_convergent_diff: JUDGEMENT $-(w_0,\ w_1)$ HAS_TYPE
pointwise_convergent

pointwise_convergent_is_pointwise_bounded: JUDGEMENT pointwise_convergent SUBTYPE_OF
pointwise_bounded

pointwise_increasing?(u): bool =
$\forall\ x$: increasing?($\lambda\ n:\ u(n)(x)$)

pointwise_decreasing?(u): bool =
  $\forall\ x$: decreasing?($\lambda\ n$: $u(n)(x)$)

$u \nearrow f$: bool = $u \longrightarrow f\ \wedge$ pointwise_increasing?(u)

$u \searrow f$: bool = $u \longrightarrow f\ \wedge$ pointwise_decreasing?(u)

plus_minus_pointwise_convergence: LEMMA
  $u \longrightarrow f \Leftrightarrow (u^+ \longrightarrow f^+ \wedge u^- \longrightarrow f^-)$

p: VAR pointwise_bounded_below

a: VAR pointwise_bounded_above

b: VAR pointwise_bounded

inf(p)(n)(x): $\mathbb{R}$ =
  inf(image[$\mathbb{N}$, $\mathbb{R}$]($\lambda\ m$: $p(m)(x)$, $\{m \mid m \geq n\}$))

limsup(b)(x): $\mathbb{R}$ =
  sup(image[$\mathbb{N}$, $\mathbb{R}$]($\lambda\ m$: inf(b)(m)(x), fullset[$\mathbb{N}$]))

sup(a)(n)(x): $\mathbb{R}$ =
  sup(image[$\mathbb{N}$, $\mathbb{R}$]($\lambda\ m$: $a(m)(x)$, $\{m \mid m \geq n\}$))

liminf(b)(x): $\mathbb{R}$ =
  inf(image[$\mathbb{N}$, $\mathbb{R}$]($\lambda\ m$: sup(b)(m)(x), fullset[$\mathbb{N}$]))

sup_inf_def: LEMMA sup(a) = $-$inf($-a$)

liminf_limsup_def: LEMMA liminf(b) = $-$limsup($-b$)

inf_pointwise_increasing: LEMMA pointwise_increasing?(inf(p))

inf_le: LEMMA inf(p)(n)(x) $\leq$ $p(n)(x)$

inf_pointwise_le: LEMMA
  $p \longrightarrow f \Rightarrow (\forall\ n, x$: inf($p$)($n$)($x$) $\leq f(x)$)

limsup_pointwise_convergence: LEMMA inf(b) $\longrightarrow$ limsup(b)

inf_pointwise_convergence_upto: LEMMA
  $p \longrightarrow f \Rightarrow$ inf(p) $\nearrow f$

pointwise_convergence_plus_minus_def: LEMMA
  $u \longrightarrow f \Rightarrow$
  (inf(u^+) $\nearrow f^+ \wedge$ inf(u^-) $\nearrow f^-$)

END pointwise_convergence

3 sup_norm

```
sup_norm[T: TYPE]: THEORY
 BEGIN

  ε: VAR ℝ>0

  c: VAR ℝ≥0

  y: VAR ℝ

  x: VAR T

  i, n: VAR ℕ

  IMPORTING reals@real_fun_ops_aux[T], reals@bounded_reals[ℝ],
            structures@const_fun_def[T, ℝ]

  bounded?(f: [T → ℝ]): bool =
      ∃ c: ∀ x: |f(x)| ≤ c

  bounded: TYPE+ = (bounded?) CONTAINING (λ x: 0)

  f, f1, f2: VAR bounded

  bounded_add: JUDGEMENT +(f1, f2) HAS_TYPE bounded

  bounded_scal: JUDGEMENT ×(y, f) HAS_TYPE bounded

  bounded_opp: JUDGEMENT −(f) HAS_TYPE bounded

  bounded_diff: JUDGEMENT −(f1, f2) HAS_TYPE bounded

  sup_norm(f): ℝ≥0 =
      IF ∃ x: TRUE
        THEN sup(extend[ℝ, ℝ≥0, bool, FALSE]({c | ∃ x: |f(x)| = c}))
      ELSE 0
      ENDIF

  sup_norm_eq_0: LEMMA
    sup_norm(f) = 0 ⇔ f = const_fun[T, ℝ](0)

  sup_norm_neg: LEMMA sup_norm(−f) = sup_norm(f)

  sup_norm_sum: LEMMA
    sup_norm(f1 + f2) ≤ sup_norm(f1) + sup_norm(f2)

  sup_norm_prop: LEMMA
    (∀ x: |f(x)| ≤ sup_norm(f)) ∧
     (∀ c: (∀ x: |f(x)| ≤ c) ⇒ sup_norm(f) ≤ c)
```

```
u: VAR sequence[bounded]

sup_norm_converges_to?(u, f): bool =
    ∀ ε:
      ∃ n: ∀ i: i ≥ n ⇒ sup_norm(u(i) − f) < ε

sup_norm_convergent?(u): bool =
    ∃ f: sup_norm_converges_to?(u, f)

sup_norm_convergent: TYPE+ = (sup_norm_convergent?) CONTAINING (λ
                                                                   n:
                                                                   λ
                                                                   x:
                                                                   0)

IMPORTING pointwise_convergence[T]

sup_norm_convergent_is_pointwise_convergent: JUDGEMENT sup_norm_convergent SUBTYPE_OF
    pointwise_convergent

sup_norm_converges_to_pointwise_convergence: LEMMA
  sup_norm_converges_to?(u, f) ⇒ u ⟶ f

END sup_norm
```

4 product_sections

```
product_sections[T1, T2: TYPE]: THEORY
 BEGIN

  X, Y: VAR set[[T1, T2]]

  a: VAR T1

  b: VAR T2

  IMPORTING topology@cross_product[T1, T2]

  x_section_emptyset: LEMMA x_section(∅, a) = ∅

  x_section_complement: LEMMA
    x_section(complement(X), a) = complement(x_section(X, a))

  x_section_union: LEMMA
    x_section((X ∪ Y), a) =
      (x_section(X, a) ∪ x_section(Y, a))

  x_section_intersection: LEMMA
    x_section((X ∩ Y), a) =
      (x_section(X, a) ∩ x_section(Y, a))

  x_section_disjoint: LEMMA
    disjoint?(X, Y) ⇒
      disjoint?(x_section(X, a), x_section(Y, a))

  y_section_emptyset: LEMMA y_section(∅, b) = ∅

  y_section_complement: LEMMA
    y_section(complement(X), b) = complement(y_section(X, b))

  y_section_union: LEMMA
    y_section((X ∪ Y), b) =
      (y_section(X, b) ∪ y_section(Y, b))

  y_section_intersection: LEMMA
    y_section((X ∩ Y), b) =
      (y_section(X, b) ∩ y_section(Y, b))

  y_section_disjoint: LEMMA
    disjoint?(X, Y) ⇒
      disjoint?(y_section(X, b), y_section(Y, b))

 END product_sections
```

Part II
Borel Sets and Functions

5 subset_algebra_def

subset_algebra_def[T: TYPE]: THEORY
BEGIN

IMPORTING sets_aux@countable_props,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sets_aux@countable_indexed_sets[T],
sets_aux@nat_indexed_sets[T], sets_aux@countable_image

n, i: VAR $\mathbb{N}$

a, b: VAR set[T]

S, X, Y: VAR setofsets[T]

NX: VAR (nonempty?[set[T]])

E: VAR sequence[set[T]]

subset_algebra_empty?(S): bool = ($\emptyset$[T] $\in S$)

subset_algebra_complement?(S): bool =
$\forall$ (x: (S)): ($\overline{x} \in S$)

subset_algebra_union?(S): bool =
$\forall$ (x, y: (S)): (($x \cup y$) $\in S$)

subset_algebra?(S): bool =
subset_algebra_empty?(S) $\wedge$
subset_algebra_complement?(S) $\wedge$ subset_algebra_union?(S)

sigma_algebra_union?(S): bool =
$\forall$ X:
is_countable[set[T]](X) $\wedge$ ($\forall$ (x: (X)): ($x \in S$)) $\Rightarrow$
($\bigcup X \in S$)

sigma_algebra?(S): bool =
subset_algebra_empty?(S) $\wedge$
subset_algebra_complement?(S) $\wedge$ sigma_algebra_union?(S)

sigma_union_implies_subset_union: LEMMA
sigma_algebra_union?(S) $\Rightarrow$ subset_algebra_union?(S)

sigma_algebra_implies_subset_algebra: LEMMA sigma_algebra?(S) $\Rightarrow$ subset_algebra?(S)

trivial_subset_algebra: (subset_algebra?) =
(singleton($\emptyset$[T]) $\cup$ singleton(fullset[T]))

subset_algebra: TYPE+ = (subset_algebra?) CONTAINING trivial_subset_algebra

sigma_algebra: TYPE+ = (sigma_algebra?) CONTAINING trivial_subset_algebra

A: VAR sigma_algebra

I: VAR set[sigma_algebra]

sigma_algebra_is_subset_algebra: JUDGEMENT sigma_algebra SUBTYPE_OF
subset_algebra

powerset_is_sigma_algebra: LEMMA
sigma_algebra?(powerset(fullset[T]))

$\mathcal{S}(X)$: sigma_algebra =
$\bigcap\{Y \mid$ sigma_algebra?(Y) $\wedge$ $(X \subseteq Y)\}$

generated_sigma_algebra_subset1: LEMMA $(X \subseteq \mathcal{S}(X))$

generated_sigma_algebra_subset2: LEMMA
$(X \subseteq Y)$ $\wedge$ sigma_algebra?(Y) $\Rightarrow$ $(\mathcal{S}(X) \subseteq Y)$

generated_sigma_algebra_idempotent: LEMMA $\mathcal{S}(A) = A$

intersection_sigma_algebra: LEMMA
$\forall$ (A, B: sigma_algebra): sigma_algebra?(($A \cap B$))

$\sigma(I)$: sigma_algebra =
$\mathcal{S}$($\bigcup$extend [setof[setof[T]], sigma_algebra, bool, FALSE](I))

sigma_member: LEMMA $(A \in I) \Rightarrow (A \subseteq \sigma(I))$

B: VAR subset_algebra

J: VAR set[subset_algebra]

powerset_is_subset_algebra: LEMMA
subset_algebra?(powerset(fullset[T]))

$\mathcal{A}(X)$: subset_algebra =
$\bigcap\{Y \mid$ subset_algebra?(Y) $\wedge$ $(X \subseteq Y)\}$

generated_subset_algebra_subset1: LEMMA $(X \subseteq \mathcal{A}(X))$

generated_subset_algebra_subset2: LEMMA
$(X \subseteq Y)$ $\wedge$ subset_algebra?(Y) $\Rightarrow$ $(\mathcal{A}(X) \subseteq Y)$

generated_subset_algebra_idempotent: LEMMA $\mathcal{A}(B) = B$

intersection_subset_algebra: LEMMA
$\forall$ (A, B: subset_algebra): subset_algebra?(($A \cap B$))

subset(J): subset_algebra =
$\mathcal{A}(\bigcup$extend [setof[setof[T]], subset_algebra, bool, FALSE](J))

subset_member: LEMMA ($B \in J$) $\Rightarrow$ ($B \subseteq$ subset(J))

finite_disjoint_union?(X)(a): bool =
$\exists$ E, n:
disjoint?(E) $\wedge$
$a = \bigcup E$ $\wedge$
($\forall$ i:
($i < n \Rightarrow (E(i) \in X)$) $\wedge$
($i \geq n \Rightarrow$ empty?($E(i)$)))

finite_disjoint_union_of?(X)(a)(E, n): bool =
disjoint?(E) $\wedge$
$a = \bigcup E$ $\wedge$
($\forall$ i:
($i < n \Rightarrow (E(i) \in X)$) $\wedge$
($i \geq n \Rightarrow$ empty?($E(i)$)))

card(X: setofsets[T], a: (finite_disjoint_union?(X))): $\mathbb{N}$ =
min({n: $\mathbb{N}$ |
$\exists$ E: finite_disjoint_union_of?(X)(a)(E, n)})

finite_disjoint_unions(X): setofsets[T] =
extend[setof[T], ((finite_disjoint_union?(X))), bool, FALSE]
(fullset[(finite_disjoint_union?(X))])

disjoint_algebra_construction: LEMMA
($\forall$ (a, b: (NX)): (($a \cap b$) $\in$ NX)) $\wedge$
($\forall$ (a: (NX)): finite_disjoint_union?(NX)($\overline{a}$))
$\Rightarrow$ $\mathcal{A}$(NX) = finite_disjoint_unions(NX)

monotone?(X): bool =
$\forall$ E:
($\forall$ n: ($E(n) \in X$)) $\Rightarrow$
((increasing?(E) $\Rightarrow$ ($\bigcup E \in X$)) $\wedge$
(decreasing?(E) $\Rightarrow$ ($\bigcap E \in X$)))

monotone_class: TYPE+ = (monotone?) CONTAINING trivial_subset_algebra

powerset_is_monotone: LEMMA monotone?(powerset(fullset[T]))

sigma_algebra_is_monotone_class: JUDGEMENT sigma_algebra SUBTYPE_OF
monotone_class

monotone_algebra_is_sigma: LEMMA
subset_algebra?(X) $\wedge$ monotone?(X) $\Rightarrow$ sigma_algebra?(X)

C: VAR monotone_class

K: VAR set[monotone_class]

monotone_class_Intersection: LEMMA
monotone?($\bigcap$extend [setof[setof[T]], monotone_class, bool, FALSE](K))

monotone_class: THEOREM $(B \subseteq C) \Rightarrow (\mathcal{S}(B) \subseteq C)$

END subset_algebra_def

6 subset_algebra

subset_algebra[T: TYPE, (IMPORTING subset_algebra_def[T]) S: subset_algebra[T]]: THEORY
BEGIN

x, y: VAR (S)

subset_algebra_emptyset: JUDGEMENT $\emptyset$[T] HAS_TYPE (S)

subset_algebra_fullset: JUDGEMENT fullset[T] HAS_TYPE (S)

subset_algebra_complement: JUDGEMENT complement(x) HAS_TYPE (S)

subset_algebra_union: JUDGEMENT union(x, y) HAS_TYPE (S)

subset_algebra_intersection: JUDGEMENT intersection(x, y) HAS_TYPE (S)

subset_algebra_difference: JUDGEMENT difference(x, y) HAS_TYPE (S)

END subset_algebra

7 sigma_algebra

sigma_algebra[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, S], sets_aux@countable_image

x, y: VAR (S)

SS: VAR sequence[(S)]

sigma_algebra_emptyset: LEMMA ($\emptyset$[T] $\in S$)

sigma_algebra_fullset: LEMMA (fullset[T] $\in S$)

sigma_algebra_complement: LEMMA ($\overline{x} \in S$)

sigma_algebra_union: LEMMA ($(x \cup y) \in S$)

sigma_algebra_intersection: LEMMA ($(x \cap y) \in S$)

sigma_algebra_difference: LEMMA ($(x \setminus y) \in S$)

sigma_algebra_IUnion: LEMMA ($\bigcup$ SS $\in S$)

sigma_algebra_IIntersection: LEMMA ($\bigcap$ SS $\in S$)

sigma_algebra_emptyset_rew: JUDGEMENT $\emptyset$[T] HAS_TYPE (S)

sigma_algebra_fullset_rew: JUDGEMENT fullset[T] HAS_TYPE (S)

sigma_algebra_complement_rew: JUDGEMENT complement(x) HAS_TYPE (S)

sigma_algebra_union_rew: JUDGEMENT union(x, y) HAS_TYPE (S)

sigma_algebra_intersection_rew: JUDGEMENT intersection(x, y) HAS_TYPE (S)

sigma_algebra_difference_rew: JUDGEMENT difference(x, y) HAS_TYPE (S)

sigma_algebra_IUnion_rew: JUDGEMENT IUnion(SS) HAS_TYPE (S)

sigma_algebra_IIntersection_rew: JUDGEMENT IIntersection(SS) HAS_TYPE (S)

END sigma_algebra

8 product_sigma_def

product_sigma_def[T_1, T_2: TYPE]: THEORY
BEGIN

IMPORTING subset_algebra_def, sigma_algebra, topology@cross_product[T_1, T_2],
product_sections[T_1, T_2], sets_aux@countable_image

i, n: VAR $\mathbb{N}$

x: VAR T_1

y: VAR T_2

X: VAR set[T_1]

Y: VAR set[T_2]

NX: VAR (nonempty?[T_1])

NY: VAR (nonempty?[T_2])

Z: VAR set[[T_1, T_2]]

S_1: VAR sigma_algebra[T_1]

S_2: VAR sigma_algebra[T_2]

measurable_rectangle?(S_1, S_2)(Z): bool =
$\exists\ X,\ Y:\ Z = X \times Y \land S_1(X) \land S_2(Y)$

measurable_rectangle(S_1, S_2): TYPE+ = (measurable_rectangle?(S_1, S_2)) CONTAINING $\emptyset$

$S_1 \times S_2$: sigma_algebra[[T_1, T_2]] =
$\mathcal{S}$(extend [setof[[T_1, T_2]], measurable_rectangle(S_1, S_2), bool, FALSE](fullset[measurable_rectangle(S_1, S_2)]

x_section_measurable: LEMMA
$(Z \in S_1 \times S_2) \Rightarrow$ (x_section(Z, x) $\in S_2$)

y_section_measurable: LEMMA
$(Z \in S_1 \times S_2) \Rightarrow$ (y_section(Z, y) $\in S_1$)

sigma_cross_projection: LEMMA
$(\mathrm{NX} \times \mathrm{NY} \in S_1 \times S_2) \Rightarrow ((\mathrm{NX} \in S_1) \land (\mathrm{NY} \in S_2))$

END product_sigma_def

9 product_sigma

```
product_sigma[T_1, T_2: TYPE, (IMPORTING subset_algebra_def) S_1: sigma_algebra[T_1],
                  S_2: sigma_algebra[T_2]]: THEORY
 BEGIN

  IMPORTING subset_algebra_def, sigma_algebra, topology@cross_product[T_1, T_2],
            product_sigma_def[T_1, T_2], sets_aux@countable_image

  n, i: VAR ℕ

  X: VAR (S_1)

  Y: VAR (S_2)

  x: VAR T_1

  y: VAR T_2

  Z: VAR set[[T_1, T_2]]

  NX: VAR (nonempty?[T_1])

  NY: VAR (nonempty?[T_2])

  R, R_1, R_2: VAR set[(measurable_rectangle?(S_1, S_2))]

  r, r_1, r_2: VAR (measurable_rectangle?(S_1, S_2))

  cross_product_is_sigma_times: LEMMA
    sigma_times(S_1, S_2)(X × Y)

  rectangle_algebra_aux: LEMMA
    𝒜(measurable_rectangle?(S_1, S_2)) =
     finite_disjoint_unions[[T_1, T_2]]
         (measurable_rectangle?(S_1, S_2))

  rectangle_algebra: subset_algebra[[T_1, T_2]] =
       finite_disjoint_unions[[T_1, T_2]]
           (measurable_rectangle?(S_1, S_2))

  rectangle_algebra_def: LEMMA
    rectangle_algebra = 𝒜(measurable_rectangle?(S_1, S_2))

  finite_disjoint_rectangles: LEMMA
    finite_disjoint_unions[[T_1, T_2]](measurable_rectangle?(S_1, S_2))(Z) ⇔
     (∃ R:
          ⋃extend [setof[[T_1, T_2]], ((measurable_rectangle?[T_1, T_2](S_1, S_2))), bool, FALSE](R)
           = Z
           ∧
           is_finite(R) ∧
```

$(\forall\ (x,\ y:\ (R)):\ x\ =\ y\ \vee$ disjoint?$(x,\ y)))$

intersection_rectangle: LEMMA
finite_disjoint_union?(measurable_rectangle?$(S_1,\ S_2)$)
$((r_1 \cap r_2))$

complement_rectangle: LEMMA
finite_disjoint_union?(measurable_rectangle?$(S_1,\ S_2)$)$(\overline{r})$

END product_sigma

10 borel

```
borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: topology]: THEORY
 BEGIN

  IMPORTING subset_algebra_def[T], topology@topology[T, S], topology@basis[T],
            sets_aux@countability

  x: VAR T

  X: VAR open

  Y: VAR closed

  Z: VAR set[T]

  B: VAR (base?[T](S))

  borel?: sigma_algebra =
      S(extend [setof[T], open[T, S], bool, FALSE](fullset[open]))

  borel: TYPE+ = (borel?) CONTAINING ∅[T]

  IMPORTING sigma_algebra[T, (borel?)]

  a, b: VAR borel

  A: VAR countable_set[borel]

  C: VAR set[borel]

  emptyset_is_borel: LEMMA borel?(∅[T])

  fullset_is_borel: LEMMA borel?(fullset[T])

  open_is_borel: LEMMA borel?(X)

  closed_is_borel: LEMMA borel?(Y)

  complement_is_borel: LEMMA borel?(ā)

  union_is_borel: LEMMA borel?((a ∪ b))

  intersection_is_borel: LEMMA borel?((a ∩ b))

  difference_is_borel: LEMMA borel?((a \ b))

  Union_is_borel: LEMMA
    borel?(⋃extend[setof[T], borel, bool, FALSE](A))

  Complement_is_borel: LEMMA
    every(borel?,
```

Complement(extend[setof[T], borel, bool, FALSE](C)))

Intersection_is_borel: LEMMA
borel?($\bigcap$extend[setof[T], borel, bool, FALSE](A))

emptyset_is_borel_judge: JUDGEMENT $\emptyset$[T] HAS_TYPE borel

fullset_is_borel_judge: JUDGEMENT fullset[T] HAS_TYPE borel

open_is_borel_judge: JUDGEMENT open SUBTYPE_OF borel

closed_is_borel_judge: JUDGEMENT closed SUBTYPE_OF borel

complement_is_borel_judge: JUDGEMENT complement(a) HAS_TYPE borel

union_is_borel_judge: JUDGEMENT union(a, b) HAS_TYPE borel

intersection_is_borel_judge: JUDGEMENT intersection(a, b) HAS_TYPE
borel

difference_is_borel_judge: JUDGEMENT difference(a, b) HAS_TYPE borel

borel_basis: LEMMA generated_sigma_algebra(B)(Z) $\Rightarrow$ borel?(Z)

borel_countable_basis: LEMMA is_countable(B) $\Rightarrow$ borel? $=$ $\mathcal{S}(B)$

END borel

11 hausdorff_borel

```
hausdorff_borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: hausdorff]: THEORY
 BEGIN

  IMPORTING topology@hausdorff_convergence[T, S], borel[T, S]

  x: VAR T

  singleton_is_borel: LEMMA borel?(singleton(x))

  singleton_is_borel_judge: JUDGEMENT singleton(x) HAS_TYPE borel

 END hausdorff_borel
```

12 borel_functions

borel_functions[(IMPORTING topology@topology_def) T_1: TYPE, S: topology[T_1], T_2: TYPE,
T: topology[T_2]]: THEORY

BEGIN

IMPORTING borel, structures@const_fun_def[T_1, T_2], topology@continuity_def[T_1, S, T_2, T],
topology@continuity[T_1, S, T_2, T]

f: VAR [$T_1 \to T_2$]

c: VAR T_2

X: VAR open[T_2, T]

B: VAR borel[T_2, T]

borel_function?(f): bool =
($\forall$ B: borel?[T_1, S](inverse_image(f, B)))

borel_function_def: LEMMA
borel_function?(f) $\equiv$
($\forall$ X: borel?[T_1, S](inverse_image[T_1, T_2](f, X)))

borel_function: TYPE = (borel_function?)

const_borel_function: LEMMA borel_function?(const_fun[T_1, T_2](c))

continuous_is_borel: JUDGEMENT continuous SUBTYPE_OF borel_function

END borel_functions

13 identity_borel

identity_borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: topology]: THEORY
BEGIN

IMPORTING borel_functions[T, S, T, S]

id_borel: LEMMA borel_function?(I[T])

I_is_borel: JUDGEMENT I[T] HAS_TYPE borel_function

END identity_borel

14 composition_borel

composition_borel[(IMPORTING topology@topology_def) T_1: TYPE, S: topology[T_1], T_2: TYPE,
T: topology[T_2], T_3: TYPE, U: topology[T_3]]: THEORY

BEGIN

IMPORTING borel_functions[T_1, S, T_2, T], borel_functions[T_2, T, T_3, U],
borel_functions[T_1, S, T_3, U]

f: VAR [$T_2 \rightarrow T_3$]

g: VAR [$T_1 \rightarrow T_2$]

composition_borel: LEMMA
borel_function?(f) $\wedge$ borel_function?(g) $\Rightarrow$
borel_function?($f \circ g$)

F: VAR borel_function[T_2, T, T_3, U]

G: VAR borel_function[T_1, S, T_2, T]

composition_is_borel: JUDGEMENT $O(F, G)$ HAS_TYPE
borel_function[T_1, S, T_3, U]

END composition_borel

15 real_borel

```
real_borel: THEORY
 BEGIN

  IMPORTING metric_space@real_topology, borel[ℝ, metric_induced_topology],
            hausdorff_borel[ℝ, metric_induced_topology]

  borel_generated_by_rational_open_interval: LEMMA
    borel? =
     S(extend [setof[ℝ], rational_open_interval, bool, FALSE](fullset[rational_open_interval]))

  borel_generated_by_open_interval: LEMMA
    borel? =
     S(extend [setof[ℝ], open_interval, bool, FALSE](fullset[open_interval]))

  open_interval_is_borel: JUDGEMENT open_interval SUBTYPE_OF borel

  closed_interval_is_borel: JUDGEMENT closed_interval SUBTYPE_OF borel

 END real_borel
```

Part III
Measures

16 generalized_measure_def

generalized_measure_def[T: TYPE, S: setofsets[T]]: THEORY
BEGIN

ASSUMING
S_empty: ASSUMPTION $S(\emptyset)$
ENDASSUMING

IMPORTING series@series, sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sets_aux@nat_indexed_sets[T],
metric_space@convergence_aux, $\overline{\mathbb{R}}_{\geq 0}$@$\overline{\mathbb{R}}_{\geq 0}$

i, j, n: VAR $\mathbb{N}$

f: VAR $[(S) \rightarrow \overline{\mathbb{R}}_{\geq 0}]$

g: VAR $[(S) \rightarrow \mathbb{R}_{\geq 0}]$

A: VAR $[\mathbb{N} \rightarrow (S)]$

a, b: VAR (S)

x: VAR set[T]

disjoint_indexed_measurable?(A): bool = disjoint?(A)

disjoint_indexed_measurable: TYPE+ = (disjoint_indexed_measurable?) CONTAINING (λ i: $\emptyset$[T])

disjoint_indexed_measurable_is_disjoint_indexed_set: JUDGEMENT disjoint_indexed_measurable SUBTYPE_OF
disjoint_indexed_set[$\mathbb{N}$, T]

X: VAR disjoint_indexed_measurable

measure_empty?(f): bool = $f(\emptyset[T])$ = (TRUE, 0)

measure_countably_additive?(f): bool =
$\forall$ X: $S(\bigcup X) \Rightarrow \sum f \circ X = f(\bigcup X)$

measure_complete?(f): bool =
($\forall$ x, a:
(($x \subseteq a$) $\wedge$ $f(a)$ = (TRUE, 0)) $\Rightarrow$ $S(x)$)

measure?(f): bool =
measure_empty?(f) $\wedge$ measure_countably_additive?(f)

complete_measure?(f): bool =
    measure?(f) $\wedge$ measure_complete?(f)

zero_measure(a): $\overline{\mathbb{R}}_{\geq 0}$ = (TRUE, 0)

measure_type: TYPE+ = (measure?) CONTAINING zero_measure

trivial_measure: measure_type =
    λ a: IF empty?(a) THEN (TRUE, 0) ELSE (FALSE, 0) ENDIF

complete_measure: TYPE+ = (complete_measure?) CONTAINING trivial_measure

complete_measure_is_measure: JUDGEMENT complete_measure SUBTYPE_OF
    measure_type

measure_disjoint_union: LEMMA
  measure?(f) $\wedge$ disjoint?(a, b) $\wedge$ $S((a \cup b))$ $\Rightarrow$
  $f((a \cup b)) = f(a) + f(b)$

finite_measure?(g): bool =
    $g(\emptyset[T])$ = 0 $\wedge$
     ($\forall$ X:
         $S(\bigcup X)$ $\Rightarrow$ series($g \circ X$) $\longrightarrow$ $g(\bigcup X)$)

complete_finite_measure?(g): bool =
    finite_measure?(g) $\wedge$
     ($\forall$ x, a: ($x \subseteq a$) $\wedge$ $g(a)$ = 0 $\Rightarrow$ $S(x)$)

trivial_finite_measure(A: (S)): [$\mathbb{R}_{\geq 0}$] = 0

finite_measure: TYPE+ = (finite_measure?) CONTAINING trivial_finite_measure

complete_finite_measure: TYPE = (complete_finite_measure?)

complete_finite_measure_is_finite_measure: JUDGEMENT complete_finite_measure SUBTYPE_OF
    finite_measure

to_measure(m: finite_measure): measure_type =
    λ a: (TRUE, $m(a)$)

F: VAR sequence[measure_type]

x_sum_measure: LEMMA measure?(λ a: ($\sum \lambda$ i : $F(i)(a)$))

END generalized_measure_def

17 measure_def

measure_def[T: TYPE, (IMPORTING subset_algebra_def[T]) S: subset_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, S], generalized_measure_def[T, S]

convergent: MACRO pred[sequence[$\mathbb{R}$]] =
convergence_sequences.convergent?;

limit: MACRO [(convergence_sequences.convergent?) $\rightarrow \mathbb{R}$] =
convergence_sequences.limit;

i, j, n: VAR $\mathbb{N}$

f: VAR [(S) $\rightarrow \overline{\mathbb{R}}_{\geq 0}$]

g: VAR [(S) $\rightarrow \mathbb{R}_{\geq 0}$]

A: VAR [$\mathbb{N} \rightarrow$ (S)]

a, b: VAR (S)

x: VAR set[T]

X: VAR disjoint_indexed_measurable

increasing_indexed_measurable?(A): bool = increasing_indexed?(A)

increasing_indexed_measurable: TYPE+ = (increasing_indexed_measurable?) CONTAINING (λ i: fullset[T])

P: VAR increasing_indexed_measurable

measure_sigma_finite?(f): bool =
$\exists$ X: $\bigcup X$ = fullset[T] $\wedge$ ($\forall$ i: $f(X(i))$‘1)

sigma_finite_measure?(f): bool =
measure?(f) $\wedge$ measure_sigma_finite?(f)

complete_sigma_finite?(f): bool =
measure?(f) $\wedge$
measure_complete?(f) $\wedge$ measure_sigma_finite?(f)

sigma_finite_measure: TYPE+ = (sigma_finite_measure?) CONTAINING zero_measure

complete_sigma_finite: TYPE = (complete_sigma_finite?)

discrete_measure: measure_type =
λ a:

```
IF is_finite(a)
  THEN (TRUE, card[T](a))
ELSE (FALSE, 0)
ENDIF
```

sigma_finite_measure_is_measure: JUDGEMENT sigma_finite_measure SUBTYPE_OF
measure_type

complete_sigma_finite_is_complete_measure: JUDGEMENT complete_sigma_finite SUBTYPE_OF
complete_measure

complete_sigma_finite_is_sigma_finite_measure: JUDGEMENT complete_sigma_finite SUBTYPE_OF
sigma_finite_measure

measure_monotone: LEMMA
$\text{measure?}(f) \wedge (a \subseteq b) \Rightarrow f(a) \leq f(b)$

measure_union: LEMMA
$\text{measure?}(f) \Rightarrow f((a \cup b)) \leq f(a) + f(b)$

measure_def: LEMMA

$$
\begin{aligned}
&(\text{measure?}(f) \Leftrightarrow \\
&\quad (\text{measure\_empty?}(f) \wedge \\
&\quad\quad (\forall\ (a,\ b\colon (S)): \\
&\quad\quad\quad \text{disjoint?}(a,\ b) \Rightarrow f((a \cup b)) = f(a) + f(b)) \\
&\quad\quad \wedge \\
&\quad\quad (\forall\ X: \\
&\quad\quad\quad S(\textstyle\bigcup X) \Rightarrow \\
&\quad\quad\quad\ f(\textstyle\bigcup X) \leq \sum f \circ X)))
\end{aligned}
$$

finite_measure_def: LEMMA

$$
\begin{aligned}
&\text{finite\_measure?}(g) \Leftrightarrow \\
&\ (g(\emptyset[T]) = 0 \wedge \\
&\quad (\forall\ (a,\ b\colon (S)): \\
&\quad\quad \text{disjoint?}(a,\ b) \Rightarrow g((a \cup b)) = g(a) + g(b)) \\
&\quad \wedge \\
&\quad (\forall\ X: \\
&\quad\quad S(\textstyle\bigcup X) \wedge \text{convergence\_sequences.convergent?}(\text{series}(g \circ X)) \Rightarrow \\
&\quad\quad\ g(\textstyle\bigcup X) \leq \\
&\quad\quad\quad \text{convergence\_sequences.limit}(\text{series}(g \circ X))))
\end{aligned}
$$

A_of(f: sigma_finite_measure): disjoint_indexed_measurable =

$$
\begin{aligned}
&\text{choose}(\{X \mid \\
&\quad\quad \textstyle\bigcup X = \text{fullset}[T] \wedge \\
&\quad\quad\ (\forall\ i\colon f(X(i))`1)\})
\end{aligned}
$$

P_of(f: sigma_finite_measure)(n): (S) =
$\bigcup \lambda\ i:$ IF $i \leq n$ THEN A_of(f)(i) ELSE $\emptyset[T]$ ENDIF

μ: VAR sigma_finite_measure

A_of_def1: LEMMA $\bigcup$A_of(μ) = fullset[T]

A_of_def2: LEMMA $\forall$ n: μ(A_of(μ)(n))‘1

P_of_def1: LEMMA $\bigcup$P_of(μ) = fullset[T]

P_of_def2: LEMMA $\forall$ n: μ(P_of(μ)(n))‘1

P_of_def3: LEMMA
  $\forall$ i, j: $i \leq j \Rightarrow$ (P_of(μ)(i) $\subseteq$ P_of(μ)(j))

sigma_finite_def1: LEMMA
  sigma_finite_measure?(f) $\Leftrightarrow$
   (measure?(f) $\wedge$
     ($\exists$ X:
        $\bigcup X$ = fullset[T] $\wedge$ ($\forall$ i: $f(X(i))$‘1)))

sigma_finite_def2: LEMMA
  sigma_finite_measure?(f) $\Leftrightarrow$
   (measure?(f) $\wedge$
     ($\exists$ P:
        $\bigcup P$ = fullset[T] $\wedge$ ($\forall$ i: $f(P(i))$‘1)))

END measure_def

18 measure_space_def

measure_space_def[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING sigma_algebra[T, S], reals@real_fun_ops_aux[T],
structures@const_fun_def[T, $\mathbb{R}$], metric_space@real_topology, topology@basis[$\mathbb{R}$],
borel[$\mathbb{R}$, metric_induced_topology], real_borel, sets_aux@countable_props,
sets_aux@inverse_image_Union, sets_aux@countable_image, sets_aux@countable_set

X: VAR set[T]

Y: VAR set[$\mathbb{R}$]

x, y, z: VAR T

f: VAR [$T \rightarrow \mathbb{R}$]

B: VAR borel

c: VAR $\mathbb{R}$

q: VAR $\mathbb{Q}$

r: VAR $\mathbb{R}_{>0}$

measurable_set?(X): bool = $S(X)$

measurable_set: TYPE+ = (measurable_set?) CONTAINING $\emptyset$[T]

a, b: VAR measurable_set

SS: VAR sequence[measurable_set]

M: VAR countable_set[(S)]

measurable_emptyset: JUDGEMENT $\emptyset$[T] HAS_TYPE measurable_set

measurable_fullset: JUDGEMENT fullset[T] HAS_TYPE measurable_set

measurable_complement: JUDGEMENT complement(a) HAS_TYPE measurable_set

measurable_union: JUDGEMENT union(a, b) HAS_TYPE measurable_set

measurable_intersection: JUDGEMENT intersection(a, b) HAS_TYPE
measurable_set

measurable_difference: JUDGEMENT difference(a, b) HAS_TYPE
measurable_set

measurable_IUnion: JUDGEMENT IUnion(SS) HAS_TYPE measurable_set

measurable_IIntersection: JUDGEMENT IIntersection(SS) HAS_TYPE
measurable_set

measurable_Union: JUDGEMENT Union(M) HAS_TYPE measurable_set

measurable_Intersection: JUDGEMENT Intersection(M) HAS_TYPE
measurable_set

measurable_function?(f): bool =
$\forall$ B: measurable_set?(inverse_image(f, B))

measurable_function: TYPE+ = (measurable_function?) CONTAINING (λ
x:
0)

g, g_1, g_2: VAR measurable_function

measurable_is_function: JUDGEMENT measurable_function SUBTYPE_OF
$[T \rightarrow \mathbb{R}]$

constant_is_measurable: JUDGEMENT (constant?$[T, \mathbb{R}]$) SUBTYPE_OF
measurable_function

U: VAR setofsets$[\mathbb{R}]$

measurable_def: LEMMA
borel? = $\mathcal{S}(U)$ $\Rightarrow$
(measurable_function?(f) $\Leftrightarrow$
($\forall$ (X: (U)): S(inverse_image(f, X))))

measurable_def2: LEMMA
measurable_function?(f) $\Leftrightarrow$
($\forall$ (i: open_interval): S(inverse_image(f, i)))

measurable_gt: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) > c\})$)

measurable_le: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) \leq c\})$)

measurable_lt: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) < c\})$)

measurable_ge: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) \geq c\})$)

measurable_gt2: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ q: $S(\{z \mid f(z) > q\})$)

measurable_le2: LEMMA

measurable_function?$(f) \Leftrightarrow (\forall\ q:\ S(\{z \mid f(z) \leq q\}))$

measurable_lt2: LEMMA
measurable_function?$(f) \Leftrightarrow (\forall\ q:\ S(\{z \mid f(z) < q\}))$

measurable_ge2: LEMMA
measurable_function?$(f) \Leftrightarrow (\forall\ q:\ S(\{z \mid f(z) \geq q\}))$

scal_measurable: JUDGEMENT $\times(c,\ g)$ HAS_TYPE measurable_function

sum_measurable: JUDGEMENT $+(g_1,\ g_2)$ HAS_TYPE measurable_function

opp_measurable: JUDGEMENT $-(g)$ HAS_TYPE measurable_function

diff_measurable: JUDGEMENT $-(g_1,\ g_2)$ HAS_TYPE measurable_function

END measure_space_def

19 measure_space

measure_space[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], reals@real_fun_ops_aux[T], power@real_fun_power[T],
real_borel,
borel_functions[$\mathbb{R}$, metric_induced_topology, $\mathbb{R}$, metric_induced_topology],
topology@constant_continuity
[$\mathbb{R}$, metric_induced_topology, $\mathbb{R}$, metric_induced_topology],
metric_space@metric_continuity
[$\mathbb{R}$, (λ (x, y: $\mathbb{R}$): $|x-y|$), $\mathbb{R}$,
(λ (x, y: $\mathbb{R}$): $|x-y|$)],
metric_space@real_continuity[$\mathbb{R}$, (λ (x, y: $\mathbb{R}$): $|x-y|$)],
pointwise_convergence[T], reals@bounded_reals[$\mathbb{R}$], finite_sets@finite_cross,
finite_sets@finite_sets_minmax_props[$\mathbb{R}$, $\leq$]

f: VAR [$T \to \mathbb{R}$]

g, g_1, g_2: VAR measurable_function[T, S]

ϕ: VAR borel_function

i, j, n, m: VAR $\mathbb{N}$

s: VAR sequence[[$T \to \mathbb{R}$]]

u: VAR sequence[measurable_function[T, S]]

x: VAR T

c, c_1, c_2, y: VAR $\mathbb{R}$

X: VAR set[T]

Y: VAR set[$\mathbb{R}$]

a: VAR $\mathbb{R}_{>0}$

borel_comp_measurable_is_measurable: JUDGEMENT $O(\phi, g)$ HAS_TYPE
measurable_function[T, S]

const_measurable: LEMMA measurable_function?(λ x: c)

nn_measurable?(f): bool =
measurable_function?(f) $\wedge$ ($\forall$ x: $0 \leq f(x)$)

nn_measurable: TYPE+ = (nn_measurable?) CONTAINING (λ x: 0)

nn_measurable_is_measurable: JUDGEMENT nn_measurable SUBTYPE_OF
measurable_function

abs_measurable: JUDGEMENT abs(g) HAS_TYPE
measurable_function[T, S]

expt_nat_measurable: JUDGEMENT expt(g, n) HAS_TYPE
measurable_function[T, S]

sq_measurable: JUDGEMENT sq(g) HAS_TYPE measurable_function[T, S]

min_measurable: JUDGEMENT min(g_1, g_2) HAS_TYPE
measurable_function[T, S]

max_measurable: JUDGEMENT max(g_1, g_2) HAS_TYPE
measurable_function[T, S]

minimum_measurable: JUDGEMENT minimum(u, n) HAS_TYPE
measurable_function[T, S]

maximum_measurable: JUDGEMENT maximum(u, n) HAS_TYPE
measurable_function[T, S]

plus_measurable: JUDGEMENT plus(g) HAS_TYPE
measurable_function[T, S]

minus_measurable: JUDGEMENT minus(g) HAS_TYPE
measurable_function[T, S]

prod_measurable: JUDGEMENT $\times$(g_1, g_2) HAS_TYPE
measurable_function[T, S]

expt_measurable: JUDGEMENT ^(g: nn_measurable, a) HAS_TYPE
measurable_function[T, S]

measurable_plus_minus: LEMMA
measurable_function?[T, S](f) $\Leftrightarrow$
(measurable_function?[T, S](f^+) $\wedge$
measurable_function?[T, S](f^-))

measurable_bounded_above?(u): bool = pointwise_bounded_above?(u)

measurable_bounded_below?(u): bool = pointwise_bounded_below?(u)

measurable_bounded?(u): bool =
measurable_bounded_above?(u) $\wedge$ measurable_bounded_below?(u)

measurable_bounded_above: TYPE+ = (measurable_bounded_above?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded_below: TYPE+ = (measurable_bounded_below?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded: TYPE+ = (measurable_bounded?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded_above_is_bounded_above: JUDGEMENT measurable_bounded_above SUBTYPE_OF
pointwise_bounded_above

measurable_bounded_below_is_bounded_below: JUDGEMENT measurable_bounded_below SUBTYPE_OF
pointwise_bounded_below

measurable_bounded_is_measurable_bounded_above: JUDGEMENT measurable_bounded SUBTYPE_OF
measurable_bounded_above

measurable_bounded_is_measurable_bounded_below: JUDGEMENT measurable_bounded SUBTYPE_OF
measurable_bounded_below

measurable_bounded_is_bounded: JUDGEMENT measurable_bounded SUBTYPE_OF
pointwise_bounded

inf_measurable: LEMMA
∀ (u: measurable_bounded_below):
measurable_function?[T, S](inf(u)(n))

sup_measurable: LEMMA
∀ (u: measurable_bounded_above):
measurable_function?[T, S](sup(u)(n))

pointwise_measurable: LEMMA
$u \longrightarrow f \Rightarrow$ measurable_function?[T, S](f)

simple?(f): bool =
measurable_function?[T, S](f) ∧
is_finite(image(f, fullset[T]))

simple: TYPE+ = (simple?) CONTAINING (λ x: 0)

simple_is_measurable: JUDGEMENT simple SUBTYPE_OF measurable_function

simple_const: LEMMA simple?(λ x: c)

nn_simple?(f): bool = (∀ x: $0 \leq f(x)$) ∧ simple?(f)

nn_simple: TYPE+ = (nn_simple?) CONTAINING (λ x: 0) nn_simple_is_simple: JUDGEMENT nn_simple SUBTYPE_OF simple

h, h_1, h_2: VAR simple

v: VAR sequence[simple]

simple_sq: JUDGEMENT sq(h) HAS_TYPE simple

simple_add: JUDGEMENT $+(h_1, h_2)$ HAS_TYPE simple

simple_scal: JUDGEMENT $\times(c, h)$ HAS_TYPE simple

simple_neg: JUDGEMENT $-(h)$ HAS_TYPE simple

simple_diff: JUDGEMENT $-(h_1, h_2)$ HAS_TYPE simple

simple_abs: JUDGEMENT abs(h) HAS_TYPE simple

simple_min: JUDGEMENT min(h_1, h_2) HAS_TYPE simple

simple_max: JUDGEMENT max(h_1, h_2) HAS_TYPE simple

simple_maximum: JUDGEMENT maximum(v, n) HAS_TYPE simple

simple_minimum: JUDGEMENT minimum(v, n) HAS_TYPE simple

simple_plus: JUDGEMENT plus(h) HAS_TYPE simple

simple_minus: JUDGEMENT minus(h) HAS_TYPE simple

simple_times: JUDGEMENT $\times(h_1, h_2)$ HAS_TYPE simple

simple_expt_nat: JUDGEMENT expt(h, n) HAS_TYPE simple

simple_expt: JUDGEMENT ˆ(h: nn_simple, a) HAS_TYPE simple

$\phi_X(x)$: $\mathbb{N}$ = IF $(x \in X)$ THEN 1 ELSE 0 ENDIF

phi_is_simple: JUDGEMENT $\phi(X: (S))$ HAS_TYPE simple

IMPORTING hausdorff_borel[$\mathbb{R}$, metric_induced_topology], partitions[T]

P: VAR finite_partition[T]

simple_def1: LEMMA
  simple?(f) $\Leftrightarrow$
    (is_finite(image(f, fullset[T])) $\wedge$
      ($\forall$ (y: (image(f, fullset[T]))):
        measurable_set?($\{x \mid y = f(x)\}$)))

constant_over?(f)(X): bool =
  $\exists\ y$: $\forall\ (x$: $(X))$: $y\ =\ f(x)$

simple_def2: LEMMA
  simple?(f) $\Leftrightarrow$
  ($\exists\ P$: every(S, P) $\wedge$ every(constant_over?(f), P))

simple_def3: LEMMA
  simple?(f) $\Leftrightarrow$
  ($\exists\ c_1,\ c_2,\ h_1,\ h_2$: $c_1 \times h_1 + c_2 \times h_2\ =\ f$)

IMPORTING sup_norm[T]

bounded_measurable?(f): bool =
  bounded?(f) $\wedge$ measurable_function?(f)

bounded_measurable: TYPE+ = (bounded_measurable?) CONTAINING (λ
x:
0)

bounded_measurable_is_bounded: JUDGEMENT bounded_measurable SUBTYPE_OF
  bounded

bounded_measurable_is_measurable: JUDGEMENT bounded_measurable SUBTYPE_OF
  measurable_function

simple_is_bounded_measurable: JUDGEMENT simple SUBTYPE_OF
  bounded_measurable

nn_bounded_measurable?(f): bool =
  bounded_measurable?(f) $\wedge$ ($\forall\ x$: $0\ \le\ f(x)$)

nn_bounded_measurable: TYPE+ = (nn_bounded_measurable?) CONTAINING (λ
x:
0)

nn_bounded_measurable_is_bounded_measurable: JUDGEMENT nn_bounded_measurable SUBTYPE_OF
  bounded_measurable

increasing_nn_simple?(u): bool =
  ($\forall\ n$: nn_simple?($u(n)$)) $\wedge$ pointwise_increasing?(u)

increasing_nn_simple: TYPE+ = (increasing_nn_simple?) CONTAINING (λ
n:
λ
x:
0)

p: VAR nn_bounded_measurable

w: VAR increasing_nn_simple

sup_norm_simple: LEMMA
  $\exists\ h$:
    $(\forall\ x:\ 0 \leq h(x)\ \&\ h(x) \leq p(x)) \wedge$
     $\text{sup\_norm}(p-h) \leq \frac{\text{sup\_norm}(p)}{2}$

nn_simple_approx(p): nn_simple =
    choose($\{h\ \mid$
      $(\forall\ x:\ 0 \leq h(x)\ \&\ h(x) \leq p(x)) \wedge$
       $\text{sup\_norm}(p-h) \leq$
        $\frac{\text{sup\_norm}(p)}{2}\})$

IMPORTING reals@sigma_nat

nn_simple_sequence(p)(n): RECURSIVE
    $\{h \mid \forall\ x:\ 0 \leq h(x)\ \&\ h(x) \leq p(x)\}$ =
  IF $n = 0$
    THEN nn_simple_approx(p)
  ELSE nn_simple_sequence(p − nn_simple_approx(p))($n-1$)
  ENDIF
   MEASURE $(\lambda\ p:\ \lambda\ n:\ n)$

nn_bounded_measurable_as_increasing_simple_sequence: LEMMA
  $\exists\ w$: sup_norm_converges_to?(w, p)

nn_bounded_measurable_as_sequence_prop: LEMMA
  sup_norm_converges_to?(w, p) $\Rightarrow$
   $(\forall\ n,\ x:\ w(n)(x) \leq p(x))$

bounded_measurable_as_increasing_sequence: LEMMA
  bounded_measurable?(f) $\Rightarrow$
   ($\exists\ v$: sup_norm_converges_to?(v, f))

nn_measurable_def: LEMMA
  $(\forall\ x:\ 0 \leq f(x)) \Rightarrow$
   (measurable_function?(f) $\Leftrightarrow$ $(\exists\ w:\ w \nearrow f)$)

measurable_as_limit_simple_def: LEMMA
  measurable_function?(f) $\Leftrightarrow$ $(\exists\ v:\ v \longrightarrow f)$

END measure_space

20 outer_measure_def

outer_measure_def[T: TYPE]: THEORY
BEGIN

IMPORTING $\overline{\mathbb{R}}_{\geq 0}$@$\overline{\mathbb{R}}_{\geq 0}$,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
sets_aux@indexed_sets_aux[$\mathbb{N}$, T]

f: VAR [set[T] $\rightarrow$ $\overline{\mathbb{R}}_{\geq 0}$]

X: VAR [$\mathbb{N}$ $\rightarrow$ set[T]]

a, b: VAR set[T]

om_empty?(f): bool = $f(\emptyset[T])$ = (TRUE, 0)

om_increasing?(f): bool =
$\forall$ a, b: $(a \subseteq b) \Rightarrow f(a) \leq f(b)$

om_countably_subadditive?(f): bool =
$\forall$ X: $f(\bigcup X) \leq \sum f \circ X$

outer_measure?(f): bool =
om_empty?(f) $\wedge$
om_increasing?(f) $\wedge$ om_countably_subadditive?(f)

zero_outer_measure(a): $\overline{\mathbb{R}}_{\geq 0}$ = (TRUE, 0)

outer_measure: TYPE+ = (outer_measure?) CONTAINING zero_outer_measure

END outer_measure_def

21 ast_def

ast_def[T: TYPE, A: (nonempty?[set[T]])]: THEORY
BEGIN

ASSUMING
IMPORTING subset_algebra_def[T]

A_empty: ASSUMPTION $A(\emptyset)$

A_fullset: ASSUMPTION A(fullset)

A_intersection: ASSUMPTION $\forall$ (a, b: (A)): $A((a \cap b))$

A_difference: ASSUMPTION
$\forall$ (a, b: (A)): finite_disjoint_union?(A)($(a \setminus b)$)
ENDASSUMING

IMPORTING generalized_measure_def[T, A], outer_measure_def[T], $\overline{\mathbb{R}}_{\geq 0}$@double_index[set[$T$]]

μ: VAR measure_type

z: VAR $\overline{\mathbb{R}}_{\geq 0}$

ε: VAR $\mathbb{R}_{>0}$

X: VAR set[T]

Y: VAR (A)

I: VAR sequence[(A)]

a, b: VAR set[T]

i, n: VAR $\mathbb{N}$

A_difference_union: LEMMA
$A(a)$ $\wedge$ finite_disjoint_union?(A)(b) $\Rightarrow$
finite_disjoint_union?(A)($(a \setminus b)$)

measure_subadditive: LEMMA
$A(\bigcup I) \Rightarrow \mu(\bigcup I) \leq \sum \mu \circ I$

generalized_monotonicity: LEMMA
disjoint?(I) $\wedge$ (IUnion(n, I) $\subseteq Y$) $\wedge$ ($\forall$ i: $i \geq n \Rightarrow$ empty?($I(i)$)) $\Rightarrow$
$\sum \mu \circ I \leq \mu(Y)$

generalized_measure_monotone: LEMMA
$\forall$ (a, b: (A), μ): $(a \subseteq b) \Rightarrow \mu(a) \leq \mu(b)$

μ^*: outer_measure =

$\lambda \ X:$
$\quad \inf(\{z \mid \exists \ I: \ \sum \mu \circ I = z \ \wedge \ (X \subseteq \bigcup I)\})$

outer_measure_eq: LEMMA $\text{ast}(\mu)(Y) = \mu(Y)$

outer_measure_def: LEMMA
$\exists \ I:$
$\quad (X \subseteq \bigcup I) \ \wedge \ \sum \mu \circ I \leq \text{ast}(\mu)(X) + \varepsilon$

END ast_def

22 outer_measure

outer_measure[T: TYPE, (IMPORTING subset_algebra_def[T]) A: subset_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, A], ast_def[T, A]

END outer_measure

23 outer_measure_props

outer_measure_props[T: TYPE, (IMPORTING outer_measure_def[T]) m: outer_measure]: THEORY
BEGIN

IMPORTING outer_measure_def[T], subset_algebra_def[T], orders@bounded_nats

i: VAR $\mathbb{N}$

x, y: VAR set[T]

A: VAR sequence[set[T]]

m_outer_empty: LEMMA $m(\emptyset[T]) = (\text{TRUE}, 0)$

m_outer_increasing: LEMMA $(x \subseteq y) \Rightarrow m(x) \leq m(y)$

m_outer_subadditive: LEMMA $m(\bigcup A) \leq \sum m \circ A$

outer_negligible?(x): bool = $m(x) = (\text{TRUE}, 0)$

outer_measurable?(x): bool =
$\forall\ y$: $m(y) = m((y \cap x)) + m((y \cap \overline{x}))$

outer_negligible: TYPE+ = (outer_negligible?) CONTAINING $\emptyset[T]$

outer_measurable: TYPE+ = (outer_measurable?) CONTAINING $\emptyset[T]$

pairwise_subadditive: LEMMA
$m(y) \leq m((y \cap x)) + m((y \cap \overline{x}))$

outer_measurable_def: LEMMA
outer_measurable?(x) $\Leftrightarrow$
($\forall\ y$:
$m((y \cap x)) + m((y \cap \overline{x})) \leq m(y)$)

outer_negligible_is_outer_measurable: JUDGEMENT outer_negligible SUBTYPE_OF
outer_measurable

a, b: VAR outer_measurable

X: VAR sequence[outer_measurable]

S: VAR setofsets[T]

outer_measurable_complement: JUDGEMENT complement(a) HAS_TYPE
outer_measurable

outer_measurable_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE outer_measurable

outer_measurable_fullset: JUDGEMENT fullset[T] HAS_TYPE
outer_measurable

outer_measurable_union: JUDGEMENT union(a, b) HAS_TYPE
outer_measurable

outer_measurable_intersection: JUDGEMENT intersection(a, b) HAS_TYPE
outer_measurable

outer_measurable_difference: JUDGEMENT difference(a, b) HAS_TYPE
outer_measurable

outer_measurable_disjoint_union: LEMMA
disjoint?(a, b) $\Rightarrow$
$m((x \cap (a \cup b))) = m((x \cap a)) + m((x \cap b))$

outer_measurable_IUnion: JUDGEMENT IUnion(X) HAS_TYPE outer_measurable

outer_measurable_IIntersection: JUDGEMENT IIntersection(X) HAS_TYPE
outer_measurable

outer_measurable_Union: LEMMA
is_countable(S) $\wedge$ every(outer_measurable?, S) $\Rightarrow$
outer_measurable?($\bigcup S$)

outer_measurable_Intersection: LEMMA
is_countable(S) $\wedge$ every(outer_measurable?, S) $\Rightarrow$
outer_measurable?($\bigcap S$)

outer_measurable_disjoint_IUnion: LEMMA
disjoint?(X) $\Rightarrow$
$m((x \cap \bigcup X)) = \sum \lambda \ i : \ m((x \cap X(i)))$

outer_measure_disjoint_IUnion: LEMMA
disjoint?(X) $\Rightarrow$ $m(\bigcup X) = \sum m \circ X$

outer_measurable_is_sigma_algebra: LEMMA
sigma_algebra?(extend[setof[T], outer_measurable, bool, FALSE]
(fullset[outer_measurable]))

induced_sigma_algebra: sigma_algebra[T] = (outer_measurable?)

IMPORTING measure_def[T, induced_sigma_algebra]

induced_measure: complete_measure =
restrict[set[T], (induced_sigma_algebra), $\overline{\mathbb{R}}_{\geq 0}$]($m$)

induced_measure_rew: LEMMA induced_measure(a) = $m(a)$

n, n_1, n_2: VAR outer_negligible

outer_negligible_emptyset: JUDGEMENT $\emptyset$[T] HAS_TYPE outer_negligible

outer_negligible_union: JUDGEMENT union(n_1, n_2) HAS_TYPE
outer_negligible

outer_negligible_subset: LEMMA $(x \subseteq n) \Rightarrow$ outer_negligible?(x)

END outer_measure_props

24 measure_props

measure_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], sigma_algebra[T, S],
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
measure_def[T, S], series@series_aux

n, i: VAR $\mathbb{N}$

a, b, M: VAR measurable_set

x, y: VAR $\overline{\mathbb{R}}_{\geq 0}$

X: VAR sequence[$\overline{\mathbb{R}}_{\geq 0}$]

DX: VAR disjoint_indexed_measurable

E: VAR sequence[measurable_set]

mu_fin?(M): bool = $m(M)$‘1

$\mu(M$: $\{m$: (S) | mu_fin?$(m)\})$: $\mathbb{R}_{\geq 0}$ = $m(M)$‘2

m_emptyset: LEMMA $m(\emptyset[T])$ = (TRUE, 0)

m_countably_additive: LEMMA $\sum m \circ \mathrm{DX} = m(\bigcup \mathrm{DX})$

m_disjoint_union: LEMMA
disjoint?(a, b) $\Rightarrow$ $m((a \cup b)) = m(a) + m(b)$

m_monotone: LEMMA $(a \subseteq b)$ $\Rightarrow$ $m(a) \leq m(b)$

m_union: LEMMA $m((a \cup b)) \leq m(a) + m(b)$

m_increasing_convergence: LEMMA
increasing?(E) $\Rightarrow$ x_converges?($m \circ E$, $m(\bigcup E)$)

m_decreasing_convergence: LEMMA
decreasing?(E) $\wedge$ mu_fin?($E(0)$) $\Rightarrow$
x_converges?($m \circ E$, $m(\bigcap E)$)

END measure_props

25 measure_theory

measure_theory[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], sigma_algebra[T, S], measure_props[T, S, m],
sets_aux@countable_indexed_sets

a, b: VAR measurable_set

X, Y: VAR set[T]

P: VAR set[T]

p: VAR pred[[$\mathbb{R}$, $\mathbb{R}$]]

f, g: VAR [$T \to \mathbb{R}$]

F, G: VAR sequence[[$T \to \mathbb{R}$]]

x: VAR T

i, j, n: VAR $\mathbb{N}$

Z: VAR setofsets[T]

null_set?(X): bool =
measurable_set?(X) $\wedge$ mu_fin?(X) $\wedge$ $\mu(X) = 0$

negligible_set?(Y): bool = $\exists$ X: null_set?(X) $\wedge$ ($Y \subseteq X$)

null_set: TYPE+ = (null_set?) CONTAINING $\emptyset$[T]

negligible: TYPE+ = (negligible_set?) CONTAINING $\emptyset$[T]

N, N_1, N_2: VAR null_set

NS: VAR sequence[null_set]

E, E_1, E_2: VAR negligible

ES: VAR sequence[negligible]

negligible_iff_measurable_null: LEMMA
(negligible_set?(X) $\wedge$ measurable_set?(X)) $\Leftrightarrow$ null_set?(X)

null_set_is_measurable: JUDGEMENT null_set SUBTYPE_OF measurable_set

null_is_negligible: JUDGEMENT null_set SUBTYPE_OF negligible null_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE null_set

null_union: JUDGEMENT union(N_1, N_2) HAS_TYPE null_set

null_intersection: JUDGEMENT intersection(N_1, N_2) HAS_TYPE null_set

null_difference: JUDGEMENT difference(N_1, N_2) HAS_TYPE null_set

null_IUnion: JUDGEMENT IUnion(NS) HAS_TYPE null_set

null_Union: LEMMA
every(null_set?, Z) $\wedge$ is_countable(Z) $\Rightarrow$ null_set?($\bigcup Z$)

negligible_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE negligible

negligible_union: JUDGEMENT union(E_1, E_2) HAS_TYPE negligible

negligible_intersection: JUDGEMENT intersection(E_1, E_2) HAS_TYPE negligible

negligible_IUnion: JUDGEMENT IUnion(ES) HAS_TYPE negligible

negligible_Union: LEMMA
every(negligible_set?, Z) $\wedge$ is_countable(Z) $\Rightarrow$ negligible_set?($\bigcup Z$)

negligible_subset: LEMMA $(X \subseteq E) \Rightarrow$ negligible_set?(X)

ae_in?(P)(X): bool =
$\exists$ E: $\forall$ (x: (X)): ($\neg$ ($x \in E$)) $\Rightarrow$ ($x \in P$)

ae?(P): bool = ae_in?(P)(fullset$[T]$)

pointwise_ae?(p)(f, g): bool =
ae?(λ x: $p(f(x), g(x))$)

ae?(p)(f, g): bool = pointwise_ae?(p)(f, g)

$f = 0$ *a.e.*: bool =
pointwise_ae?(restrict[[number, number], [$\mathbb{R}$, $\mathbb{R}$], boolean](=))
(f, λ x: 0)

$f \geq 0$ *a.e.*: bool = pointwise_ae?($\leq$)((λ x: 0), f)

$f > 0$ *a.e.*: bool = pointwise_ae?($<$)((λ x: 0), f)

$f \leq g$ *a.e.*: bool = pointwise_ae?($\leq$)(f, g)

$f = g$ *a.e.*: bool =
pointwise_ae?(restrict[[number, number], [$\mathbb{R}$, $\mathbb{R}$], boolean](=))
(f, g)

ae_eq_equivalence: LEMMA equivalence?(ae_eq?)

ae_le_reflexive: LEMMA reflexive?(ae_le?)

ae_le_antisymmetric: LEMMA
  $f \leq g$ *a.e.* $\wedge$ $g \leq f$ *a.e.* $\Rightarrow$ $f = g$ *a.e.*

ae_le_transitive: LEMMA transitive?(ae_le?)

ae_convergence_in?$(X)(F, f)$: bool =
  ae_in?(λ x: λ n: $F(n)(x) \longrightarrow f(x)$)$(X)$

ae_cauchy_in?$(X)(F)$: bool =
  ae_in?(λ x: cauchy?(λ n: $F(n)(x)$))(X)

$F \longrightarrow f$ *a.e.*: bool = ae_convergence_in?(fullset$[T]$)(F, f)

ae_cauchy?(F): bool = ae_cauchy_in?(fullset$[T]$)(F)

ae_convergence_cauchy: LEMMA $F \longrightarrow f$ *a.e.* $\Rightarrow$ ae_cauchy?(F)

ae_convergence_eq: LEMMA
  $F \longrightarrow f$ *a.e.* $\Rightarrow$ ($F \longrightarrow g$ *a.e.* $\Leftrightarrow$ $f = g$ *a.e.*)

ae_eq_convergence: LEMMA
  $F \longrightarrow f$ *a.e.* $\wedge$ ($\forall$ n: $F(n) = G(n)$ *a.e.*) $\Rightarrow$
  $G \longrightarrow f$ *a.e.*

increasing?(F) *a.e.*: bool =
  $\exists$ E:
    $\forall$ x:
      $\neg$ $(x \in E)$ $\Rightarrow$
      ($\forall$ i, j: $i \leq j \Rightarrow F(i)(x) \leq F(j)(x)$)

decreasing?(F) *a.e.*: bool =
  $\exists$ E:
    $\forall$ x:
      $\neg$ $(x \in E)$ $\Rightarrow$
      ($\forall$ i, j: $i \leq j \Rightarrow F(j)(x) \leq F(i)(x)$)

ae_monotonic_converges?(F, f): bool =
  $F \longrightarrow f$ *a.e.* $\wedge$ (increasing?(F) *a.e.* $\vee$ decreasing?(F) *a.e.*)

ae_convergent?(F): bool = $\exists$ f: $F \longrightarrow f$ *a.e.*

END measure_theory

26 monotone_classes

monotone_classes[T: TYPE, C: (nonempty?[set[T]])]: THEORY
BEGIN

IMPORTING subset_algebra_def[T]

a, b: VAR (C)

x: VAR $(\mathcal{S}(C))$

IMPORTING measure_def[T, $(\mathcal{S}(C))$], sigma_algebra, measure_props

monotone_finite_measures: COROLLARY
$\forall$ (ν, μ: finite_measure):
($\forall$ a, b: $((a \cap b) \in C)$) $\wedge$
($\forall$ a: finite_disjoint_union?$(C)(\overline{a})$) $\wedge$ ($\forall$ a: $\mu(a) = \nu(a)$)
$\Rightarrow$ ($\forall$ x: $\mu(x) = \nu(x)$)

END monotone_classes

27 hahn_kolmogorov

hahn_kolmogorov[T: TYPE, (IMPORTING subset_algebra_def[T]) A: subset_algebra,
(IMPORTING measure_def[T, A]) μ: measure_type]: THEORY
BEGIN

IMPORTING outer_measure[T, A], outer_measure_props[T, μ^*]

x: VAR (A)

algebra_in_induced_sigma_algebra: LEMMA ($A \subseteq$ induced_sigma_algebra)

IMPORTING measure_theory[T, induced_sigma_algebra, induced_measure]

induced_measure_measure: LEMMA induced_measure(x) = $\mu(x)$

END hahn_kolmogorov

Part IV
Finite Measures

28 finite_measure

finite_measure[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: finite_measure]: THEORY
BEGIN

IMPORTING sets_aux@sets_lemmas_aux, sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sigma_algebra[T, S],
series@series_aux,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]]

X: VAR $[\mathbb{N} \to (S)]$

A, B: VAR (S)

fm_emptyset: LEMMA $\mu(\emptyset) = 0$

fm_convergence: LEMMA
disjoint?(X) $\Rightarrow$ series($\mu \circ X$) $\longrightarrow \mu(\bigcup X)$

fm_disjointunion: LEMMA
disjoint?(A, B) $\Rightarrow$
$\mu((A \cup B)) = \mu(A) + \mu(B)$

fm_complement: LEMMA $\mu(\overline{A}) = \mu(\text{fullset}) - \mu(A)$

fm_union: LEMMA
$\mu((A \cup B)) =$
$\mu(A) + \mu(B) - \mu((A \cap B))$

fm_intersection: LEMMA
$\mu((A \cap B)) =$
$\mu(A) + \mu(B) - \mu((A \cup B))$

fm_difference: LEMMA
$\mu((A \setminus B)) =$
$\mu(A) - \mu(B) + \mu((B \setminus A))$

fm_subset: LEMMA
$(A \subseteq B) \Rightarrow \mu(B) = \mu(A) + \mu((B \setminus A))$

fm_subset_le: LEMMA $(A \subseteq B) \Rightarrow \mu(A) \leq \mu(B)$

fm_monotone: LEMMA $(A \subseteq B) \Rightarrow \mu(A) \leq \mu(B)$

fm_IUnion: LEMMA increasing?(X) $\Rightarrow \mu \circ X \longrightarrow \mu(\bigcup X)$

fm_IIntersection: LEMMA
  decreasing?(X) $\Rightarrow$ $\mu \circ X \longrightarrow \mu(\bigcap X)$

IMPORTING measure_def[T, S]

measure_from: measure_type = λ A: (TRUE, $\mu(A)$)

END finite_measure

Part V
Complete Measures

29 complete_measure_theory

complete_measure_theory[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: complete_measure]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], sigma_algebra[T, S], measure_theory[T, S, μ],
measure_props[T, S, μ]

N: VAR null_set

X: VAR set[T]

E: VAR negligible

f: VAR $[T \to \mathbb{R}]$

g: VAR measurable_function

null_subset: LEMMA $(X \subseteq N) \Rightarrow$ null_set?(X)

null_is_negligible: LEMMA null_set?(X) $\Leftrightarrow$ negligible_set?(X)

ae_eq_measurable: LEMMA $f = g$ *a.e.* $\Rightarrow$ measurable_function?(f)

END complete_measure_theory

30 measure_completion_aux

measure_completion_aux[T: TYPE]: THEORY
BEGIN

IMPORTING subset_algebra_def[T], measure_def, measure_theory, measure_props

XS: VAR setofsets[T]

A, B, X: VAR set[T]

z: VAR $\overline{\mathbb{R}}_{\geq 0}$

almost_measurable?(S: sigma_algebra[T], m: measure_type[T, S])(X): bool =
$\exists$ (Y: (S), N_1, N_2: negligible[T, S, m]):
$X = ((Y \cup N_1) \setminus N_2)$

empty_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
almost_measurable?(S, m)($\emptyset$[T])

complement_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
almost_measurable?(S, m)(X) $\Leftrightarrow$
almost_measurable?(S, m)($\overline{X}$)

Union_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
every(almost_measurable?(S, m), XS) $\wedge$ is_countable(XS) $\Rightarrow$
almost_measurable?(S, m)($\bigcup$XS)

completion(S: sigma_algebra[T], m: measure_type[T, S]): sigma_algebra[T] =
{X | almost_measurable?(S, m)(X)}

generated_completion: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
$\mathcal{S}$(($S \cup$ extend [setof[T], negligible[T, S, m], bool, FALSE](fullset[negligible[T, S, m]])))
= completion(S, m)

completion_extends: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
$S(X) \Rightarrow$ completion(S, m)(X)

negligible_completion: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
negligible_set?[T, S, m](X) $\Rightarrow$ completion(S, m)(X)

is_completion(S: sigma_algebra[T], m: measure_type[T, S])(A, B): bool =
completion(S, m)(A) $\wedge$ $S(B)$ $\Rightarrow$
($\exists$ (N_1, N_2: negligible[T, S, m]):
$A = ((B \cup N_1) \setminus N_2)$)

m_completions: LEMMA
  ∀ (S: sigma_algebra[T], m: measure_type[T, S], X, A, B):
    completion(S, m)(X) ∧
     S(A) ∧
      S(B) ∧ is_completion(S, m)(X, A) ∧ is_completion(S, m)(X, B)
     ⇒ m(A) = m(B)

choose_completion: LEMMA
  ∀ (S: sigma_algebra[T], m: measure_type[T, S], X):
    completion(S, m)(X) ⇒
     is_completion(S, m)
                  (X,
                    choose({Y: (S) |
                              ∃ (N_1, N_2: negligible[T, S, m]):
                                X = ((Y ∪ N_1) \ N_2)}))

completion(S: sigma_algebra[T], m: measure_type[T, S]):
      complete_measure[T, completion(S, m)] =
    λ (X: (completion(S, m))):
      m(choose({Y: (S) |
                   ∃ (N_1, N_2: negligible[T, S, m]):
                     X = ((Y ∪ N_1) \ N_2)}))

completion_measurable: LEMMA
  ∀ (S: sigma_algebra[T], m: measure_type[T, S], X: (S)):
    completion(S, m)(X) = m(X)

completion_negligible: LEMMA
  ∀ (S: sigma_algebra[T], m: measure_type[T, S], N: negligible[T, S, m]):
    completion(S, m)(N) = (TRUE, 0)

END measure_completion_aux

31 measure_completion

```
measure_completion[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
                   (IMPORTING measure_def[T, S]) m: measure_type]: THEORY
 BEGIN

  IMPORTING measure_completion_aux[T], measure_theory[T, S, m]

  X: VAR (S)

  N: VAR negligible[T, S, m]

  sigma_algebra_completion: sigma_algebra[T] = completion(S, m)

  generated_completion: LEMMA
    𝒮((S ∪ extend [setof[T], negligible[T, S, m], bool, FALSE](fullset[negligible[T, S, m]])))
      = sigma_algebra_completion

  completion: complete_measure[T, completion(S, m)] =
        completion(S, m)

  completion_measurable: LEMMA completion(X) = m(X)

  completion_negligible: LEMMA completion(N) = (TRUE, 0)

 END measure_completion
```

Part VI

Integration

32 isf

isf[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_theory[T, S, m], measure_props[T, S, m]

x: VAR T

f: VAR $[T \rightarrow \mathbb{R}]$

g: VAR measurable_function

X: VAR (S)

Y: VAR set[T]

nonzero_set?(f): set[T] $= \{x \mid f(x) \neq 0\}$

nonzero_measurable: LEMMA measurable_set?(nonzero_set?(g))

nonzero_set_phi: LEMMA nonzero_set?(ϕ_X) $= X$

isf?(f): bool = simple?(f) $\wedge$ mu_fin?(nonzero_set?(f))

isf_zero: LEMMA isf?(λ x: 0)

isf: TYPE+ = (isf?) CONTAINING (λ x: 0)

isf_is_simple: JUDGEMENT isf SUBTYPE_OF simple

i, i_1, i_2: VAR isf

w: VAR sequence[isf]

c: VAR $\mathbb{R}$

n: VAR $\mathbb{N}$

pn: VAR $\mathbb{N}_{>0}$

E: VAR (mu_fin?)

h: VAR simple

nnx: VAR $\mathbb{R}_{\geq 0}$

isf_add: JUDGEMENT $+(i_1,\ i_2)$ HAS_TYPE isf

isf_scal: JUDGEMENT $\times(c,\ i)$ HAS_TYPE isf

isf_opp: JUDGEMENT $-(i)$ HAS_TYPE isf

isf_diff: JUDGEMENT $-(i_1,\ i_2)$ HAS_TYPE isf

isf_abs: JUDGEMENT abs(i) HAS_TYPE isf

isf_min: JUDGEMENT min(i_1, i_2) HAS_TYPE isf

isf_max: JUDGEMENT max(i_1, i_2) HAS_TYPE isf

isf_minimum: JUDGEMENT minimum(w, n) HAS_TYPE isf

isf_maximum: JUDGEMENT maximum(w, n) HAS_TYPE isf

isf_plus: JUDGEMENT plus(i) HAS_TYPE isf

isf_minus: JUDGEMENT minus(i) HAS_TYPE isf

isf_sq: JUDGEMENT sq(i) HAS_TYPE isf

isf_prod: JUDGEMENT $\times(i_1,\ i_2)$ HAS_TYPE isf

isf_phi: JUDGEMENT $\phi(E)$ HAS_TYPE isf

isf_expt: JUDGEMENT expt(i, pn) HAS_TYPE isf

isf_times_simple_is_isf: JUDGEMENT $\times(i,\ h)$ HAS_TYPE isf

P: VAR pred[isf]

isf_induction: LEMMA
$(P(\lambda\ x:\ 0) \land (\forall\ c,\ E,\ i:\ P(i) \Rightarrow P(c \times \phi_E + i))) \Rightarrow$
$P(i)$

p, p_1, p_2: VAR finite_partition[T]

finite_partition_of?(f)(p): bool =
∀ (E: (p)):
$S(E)$ ∧
constant_over?(f)(E) ∧
(empty?(E) ∨ f(choose(E)) = 0 ∨ mu_fin?(E))

isf_def: LEMMA
isf?(f) ⇔ (∃ (p: (finite_partition_of?(f))): TRUE)

IMPORTING sigma_set@sigma_countable

$\text{isf\_integral}(i):\ \mathbb{R}\ =$
$\quad \sum_{\text{image}[T,\ \mathbb{R}](i,\ \text{fullset}[T])} \lambda\ c:\ \text{IF}\ c\ =\ 0\ \text{THEN}\ 0\ \text{ELSE}\ c \times \mu(\text{inverse\_image}\ [T,\ \mathbb{R}](i,\ \text{singleton}\ [\mathbb{R}](c))$

$\text{isf\_integral\_phi}:\ \text{LEMMA}\ \ \text{isf\_integral}(\phi_E)\ =\ \mu(E)$

$\text{isf\_integral\_zero}:\ \text{LEMMA}\ \ \text{isf\_integral}(\lambda\ x:\ 0)\ =\ 0$

$\text{isf\_integral\_def}:\ \text{LEMMA}$
$\quad \text{finite\_partition\_of?}(i)(p)\ \Rightarrow$
$\quad\quad \text{isf\_integral}(i)\ =$
$\quad\quad\quad \text{LET}\ f\ =$
$\quad\quad\quad\quad\quad \lambda\ Y:$
$\quad\quad\quad\quad\quad\quad \text{IF}\ (\neg\ p(Y))\ \vee\ \text{empty?}(Y)\ \vee\ i(\text{choose}[T](Y))\ =\ 0$
$\quad\quad\quad\quad\quad\quad\quad \text{THEN}\ 0$
$\quad\quad\quad\quad\quad\quad \text{ELSE}\ i(\text{choose}[T](Y)) \times \mu(Y)$
$\quad\quad\quad\quad\quad\quad \text{ENDIF}$
$\quad\quad\quad\quad \text{IN}\ \sum_p f$

$\text{isf\_integral\_scal}:\ \text{LEMMA}$
$\quad \text{isf\_integral}(c \times i)\ =\ c \times \text{isf\_integral}(i)$

$\text{isf\_integral\_opp}:\ \text{LEMMA}$
$\quad \text{isf\_integral}(-i)\ =\ -\text{isf\_integral}(i)$

$\text{isf\_integral\_add}:\ \text{LEMMA}$
$\quad \text{isf\_integral}(i_1 + i_2)\ =$
$\quad\quad \text{isf\_integral}(i_1) + \text{isf\_integral}(i_2)$

$\text{isf\_integral\_diff}:\ \text{LEMMA}$
$\quad \text{isf\_integral}(i_1 - i_2)\ =$
$\quad\quad \text{isf\_integral}(i_1) - \text{isf\_integral}(i_2)$

$\text{isf\_integral\_pos}:\ \text{LEMMA}$
$\quad (\forall\ x:\ i(x)\ \geq\ 0)\ \Rightarrow\ \text{isf\_integral}(i)\ \geq\ 0$

$\text{isf\_integral\_le}:\ \text{LEMMA}$
$\quad (\forall\ x:\ i_1(x)\ \leq\ i_2(x))\ \Rightarrow$
$\quad\quad \text{isf\_integral}(i_1)\ \leq\ \text{isf\_integral}(i_2)$

$\text{isf\_integral\_abs}:\ \text{LEMMA}$
$\quad |\text{isf\_integral}(i)|\ \leq\ \text{isf\_integral}(|i|)$

$\text{isf\_bounded}:\ \text{LEMMA}$
$\quad \exists\ \text{nnx}:\ \forall\ x:\ -\text{nnx}\ \leq\ i(x)\ \wedge\ i(x)\ \leq\ \text{nnx}$

$\text{isf\_integral\_bound}:\ \text{LEMMA}$
$\quad (\forall\ x:\ |i(x)|\ \leq\ \text{nnx})\ \Rightarrow$
$\quad\quad \text{isf\_integral}(|i|)\ \leq\ \text{nnx} \times \mu(\text{nonzero\_set?}(i))$

$\text{isf\_ae\_0}:\ \text{LEMMA}$

(simple?(f) $\wedge$ $f = 0$ *a.e.*) $\Leftrightarrow$
(isf?(f) $\wedge$ isf_integral($|f|$) = 0)

isf_ae_eq: LEMMA
$i_1 = i_2$ *a.e.* $\Rightarrow$ isf_integral(i_1) = isf_integral(i_2)

isf_ae_0_le: LEMMA $i \geq 0$ *a.e.* $\Rightarrow$ $0 \leq$ isf_integral(i)

isf_ae_le: LEMMA
$i_1 \leq i_2$ *a.e.* $\Rightarrow$ isf_integral(i_1) $\leq$ isf_integral(i_2)

isf_ae_ge_0: LEMMA $i \geq 0$ *a.e.* $\wedge$ isf_integral(i) = 0 $\Rightarrow$ $i = 0$ *a.e.*

u: VAR increasing_nn_simple

isf_convergence: LEMMA
$u \nearrow i$ $\Rightarrow$ (isf_integral $\circ$ u) $\nearrow$ isf_integral(i)

END isf

33 nn_integral

nn_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_props[T, S, m], measure_theory[T, S, m],
isf[T, S, m]

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

limit: MACRO [convergent → $\mathbb{R}$] = topological_convergence.limit

n: VAR $\mathbb{N}$

pn: VAR $\mathbb{N}_{>0}$

x: VAR T

c: VAR $\mathbb{R}_{\geq 0}$

E: VAR measurable_set

F: VAR (mu_fin?)

g: VAR measurable_function

h: VAR nn_bounded_measurable

nn_isf?(i: isf): bool = $\forall$ x: $i(x) \geq 0$

nn_isf: TYPE+ = (nn_isf?) CONTAINING (λ x: 0)

i: VAR nn_isf

w: VAR sequence[nn_isf]

increasing_nn_isf?(u: sequence[nn_isf]): bool =
pointwise_increasing?(u)

increasing_nn_isf: TYPE+ = (increasing_nn_isf?) CONTAINING (λ
n:
λ
x:
0)

u, u_1, u_2: VAR increasing_nn_isf

nn_integrable?(g: [T → $\mathbb{R}_{\geq 0}$]): bool =
$\exists$ u:

$u \longrightarrow g \ \wedge$
topological_convergence.convergent?(isf_integral $\circ$ u)

nn_integrable_zero: LEMMA nn_integrable?(λ x: 0)

nn_integrable: TYPE+ = (nn_integrable?) CONTAINING (λ x: 0)

f, f_1, f_2: VAR nn_integrable

nn_integrable_is_nonneg: LEMMA $f(x) \geq 0$

nn_integrable_is_measurable: JUDGEMENT nn_integrable SUBTYPE_OF measurable_function

nn_convergence: LEMMA
$u_1 \longrightarrow f \ \wedge$
$u_2 \longrightarrow f \ \wedge$ topological_convergence.convergent?(isf_integral $\circ$ u_1)
$\Rightarrow$
(topological_convergence.convergent?(isf_integral $\circ$ u_2) $\wedge$
topological_convergence.limit(isf_integral $\circ$ u_1) =
topological_convergence.limit(isf_integral $\circ$ u_2))

nn_integral(f): $\mathbb{R}_{\geq 0}$ =
topological_convergence.limit
(isf_integral $\circ$ choose($\{u \mid u \longrightarrow f\}$))

nn_integrable_add: JUDGEMENT $+(f_1, f_2)$ HAS_TYPE nn_integrable

nn_integrable_scal: JUDGEMENT $\times(c, f)$ HAS_TYPE nn_integrable

nn_isf_is_nn_integrable: JUDGEMENT nn_isf SUBTYPE_OF nn_integrable

nn_integral_isf: LEMMA nn_integral(i) = isf_integral(i)

nn_integrable_le: LEMMA
$(\forall \ x\colon \ 0 \leq g(x) \wedge g(x) \leq f(x)) \Rightarrow$
(nn_integrable?(g) $\wedge$ nn_integral(g) $\leq$ nn_integral(f))

nn_integral_zero: LEMMA nn_integral(λ x: 0) = 0

nn_integral_phi: LEMMA nn_integral(ϕ_F) = $\mu(F)$

nn_integral_add: LEMMA
nn_integral($f_1 + f_2$) =
nn_integral(f_1) + nn_integral(f_2)

nn_integral_scal: LEMMA
nn_integral($c \times f$) = $c \times$ nn_integral(f)

nn_integrable_prod: JUDGEMENT $\times(f, h)$ HAS_TYPE nn_integrable nn_indefinite_integrable: LEMMA nn_integrable?($\phi_E \times f$)

nn_0_le: LEMMA $0 \leq$ nn_integral(f)

nn_integral_def: LEMMA
$\exists\ u$:
$u \longrightarrow f \ \wedge$ isf_integral $\circ\, u \longrightarrow$ nn_integral(f)

END nn_integral

34 integral

$\int$[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_theory[T, S, m], nn_integral[T, S, m]

g, g_1, g_2, g_3, g_4: VAR nn_integrable

x: VAR T

integrable?(f: [$T \to \mathbb{R}$]): bool =
$\exists$ (g, h: nn_integrable): $f = g - h$

integrable: TYPE+ = (integrable?) CONTAINING (λ x: 0)

nn_integrable_is_integrable: JUDGEMENT nn_integrable SUBTYPE_OF
integrable

isf_is_integrable: JUDGEMENT isf SUBTYPE_OF integrable

integrable_is_measurable: JUDGEMENT integrable SUBTYPE_OF
measurable_function

f, f_1, f_2: VAR integrable

w: VAR sequence[integrable]

f_0: VAR [$T \to \mathbb{R}$]

h: VAR measurable_function

ε: VAR $\mathbb{R}_{>0}$

c: VAR $\mathbb{R}$

nnc: VAR $\mathbb{R}_{\geq 0}$

E: VAR measurable_set

F: VAR (mu_fin?)

i: VAR isf

n: VAR $\mathbb{N}$

integrable_equiv: LEMMA
$g_1 - g_3 = g_2 - g_4 \Rightarrow$
nn_integral(g_1) − nn_integral(g_3) =
nn_integral(g_2) − nn_integral(g_4)

integrable_add: JUDGEMENT $+(f_1,\ f_2)$ HAS_TYPE integrable

integrable_scal: JUDGEMENT $\times(c,\ f)$ HAS_TYPE integrable

integrable_opp: JUDGEMENT $-(f)$ HAS_TYPE integrable

integrable_diff: JUDGEMENT $-(f_1,\ f_2)$ HAS_TYPE integrable

integrable_zero: LEMMA integrable?($\lambda\ x:\ 0$)

integrals(f): set[$\mathbb{R}$] =
    {c |
        $\exists$ (g, h: nn_integrable):
          $f = g - h \wedge$
           c = nn_integral(g) − nn_integral(h)}

nonempty_integrals: LEMMA nonempty?[$\mathbb{R}$](integrals(f))

singleton_integrals: LEMMA singleton?[$\mathbb{R}$](integrals(f))

$\int f:\ \mathbb{R}$ = choose[$\mathbb{R}$](integrals(f))

nn_integrable_is_nn_integrable: LEMMA
  $(\forall\ x:\ f(x) \geq 0) \Rightarrow$ nn_integrable?(f)

integral_nn: LEMMA $\int g$ = nn_integral(g)

integral_zero: LEMMA $\int \lambda\ x:\ 0 = 0$

integral_phi: LEMMA $\int \phi_F = \mu(F)$

integral_add: LEMMA $\int f_1 + f_2 = \int f_1 + \int f_2$

integral_scal: LEMMA $\int c \times f = c \times \int f$

integral_opp: LEMMA $\int -f = -\int f$

integral_diff: LEMMA $\int f_1 - f_2 = \int f_1 - \int f_2$

integral_nonneg: LEMMA $(\forall\ x:\ f(x) \geq 0) \Rightarrow \int f \geq 0$

integrable_abs: JUDGEMENT abs(f) HAS_TYPE integrable

integrable_max: JUDGEMENT max(f_1, f_2) HAS_TYPE integrable

integrable_min: JUDGEMENT min(f_1, f_2) HAS_TYPE integrable

integrable_plus: JUDGEMENT plus(f) HAS_TYPE integrable

integrable_minus: JUDGEMENT minus(f) HAS_TYPE integrable integral_abs: LEMMA $\left|\int f\right| \;\leq\; \int |f|$

integrable_pm_def: LEMMA
  integrable?(f_0) $\Leftrightarrow$
   (integrable?(${f_0}^{+}$) $\wedge$ integrable?(${f_0}^{-}$))

integral_pm: LEMMA $\int f \;=\; \int f^{+} - \int f^{-}$

integrable_abs_def: LEMMA integrable?($|h|$) $\Leftrightarrow$ integrable?(h)

integrable_nz_finite: LEMMA
  measurable_set?($\{x \mid |f(x)| \;\geq\; \varepsilon\}$) $\wedge$
   mu_fin?($\{x \mid |f(x)| \;\geq\; \varepsilon\}$)

isf_integral: LEMMA $\int i \;=\;$ isf_integral(i)

integral_ae_eq: LEMMA
  $f = h$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int f \;=\; \int h$)

integral_prod: LEMMA
  $|h| \leq \lambda\ x :$ nnc *a.e.* $\Rightarrow$
   (integrable?($f \times h$) $\wedge$
     $\int |f \times h| \;\leq\;$ nnc $\times \int |f|$)

indefinite_integrable: LEMMA integrable?($\phi_E \times f$)

integral_ae_le: LEMMA $f_1 \leq f_2$ *a.e.* $\Rightarrow$ $\int f_1 \;\leq\; \int f_2$

integral_ae_abs: LEMMA
  $|h| \leq |f|$ *a.e.* $\Rightarrow$
   (integrable?(h) $\wedge$ $\left|\int h\right| \;\leq\; \int |f|$)

bounded_is_indefinite_integrable: LEMMA
  bounded?($\phi_F \times h$) $\Rightarrow$
   (integrable?($\phi_F \times h$) $\wedge$
     $\left|\int \phi_F \times h\right| \;\leq$
      $\mu(F) \times$ sup_norm($\phi_F \times h$))

integral_abs_0: LEMMA $\int |f| \;=\; 0 \Rightarrow f = 0$ *a.e.*

measurable_ae_0: LEMMA
  $h = 0$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int h \;=\; 0$)

integral_ae_ge_0: LEMMA $f \geq 0$ *a.e.* $\wedge$ $\int f \;=\; 0 \Rightarrow f = 0$ *a.e.*

integrable_maximum: JUDGEMENT maximum(w, n) HAS_TYPE integrable

integrable_minimum: JUDGEMENT minimum(w, n) HAS_TYPE integrable

integrable_split: LEMMA

$\forall\ (h\colon [T \to \mathbb{R}])$:
  integrable?$(h) \Leftrightarrow$
    integrable?$(\phi_E \times h) \ \wedge$
      integrable?$(\phi_{\overline{E}} \times h)$

integral_split: LEMMA
  $\int f =$
    $\int \phi_E \times f + \int \phi_{\overline{E}} \times f$

END $\int$

35 finite_integral

finite_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: finite_measure]: THEORY

BEGIN

IMPORTING $\int$[T, S, to_measure(μ)]

bounded_measurable_is_integrable: JUDGEMENT bounded_measurable SUBTYPE_OF integrable

END finite_integral

36 integral_convergence_scaf

integral_convergence_scaf[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_theory[T, S, m], $\int$[T, S, m]

f: VAR $[T \to \mathbb{R}]$

F: VAR sequence[integrable]

monotone_convergence_scaf: LEMMA
$F \nearrow f \ \wedge$ bounded?$(\int \circ F) \Rightarrow$
(integrable?$(f) \ \wedge \ (\int \circ F) \nearrow \int f$)

END integral_convergence_scaf

37 integral_convergence

integral_convergence[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING integral_convergence_scaf[T, S, m]

i, j, n: VAR $\mathbb{N}$

f, g: VAR integrable

F: VAR sequence[integrable]

E: VAR negligible

x: VAR T

monotone_convergence: THEOREM
increasing?(F) $a.e.$ $\Rightarrow$
((($\exists\ f$: $F \longrightarrow f$ $a.e.$) $\Leftrightarrow$ bounded?($\int \circ F$)) $\wedge$
($\forall\ f$: $F \longrightarrow f$ $a.e.$ $\Rightarrow$ ($\int \circ F$) $\nearrow \int f$))

dominated_convergence: THEOREM
($\forall\ n$: $|F(n)| \leq f$ $a.e.$) $\wedge$ ae_convergent?(F) $\Rightarrow$
($\exists\ g$: $F \longrightarrow g$ $a.e.$ $\wedge$ $\int \circ F \longrightarrow \int g$)

END integral_convergence

38 complete_integral

complete_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: complete_measure]: THEORY

BEGIN

IMPORTING complete_measure_theory[T, S, μ], $\int$[T, S, μ],
integral_convergence[T, S, μ]

f: VAR integrable

h: VAR $[T \rightarrow \mathbb{R}]$

F: VAR sequence[integrable]

n: VAR $\mathbb{N}$

x: VAR T

complete_integral_ae_eq: LEMMA
$f = h$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int f = \int h$)

complete_measurable_ae_0: LEMMA
$h = 0$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int h = 0$)

monotone_convergence_complete: THEOREM
ae_monotonic_converges?(F, h) $\wedge$ bounded?($\int \circ F$) $\Rightarrow$
(integrable?(h) $\wedge$
monotonic_converges?(($\int \circ F$), $\int h$))

dominated_convergence_complete: THEOREM
($\forall$ n: $|F(n)| \leq f$ *a.e.*) $\wedge$ $F \longrightarrow h$ *a.e.* $\Rightarrow$
(integrable?(h) $\wedge$ $\int \circ F \longrightarrow \int h$)

END complete_integral

39 indefinite_integral

indefinite_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_props[T, S, m], $\int$[T, S, m], integral_convergence[T, S, m]

f, f_1, f_2: VAR integrable

g: VAR $[T \to \mathbb{R}]$

h: VAR measurable_function[T, S]

DX: VAR disjoint_indexed_measurable

E, E_1, E_2: VAR measurable_set

F: VAR (mu_fin?)

N: VAR null_set

c: VAR $\mathbb{R}$

x: VAR T

n: VAR $\mathbb{N}$

integrable?(E)(g): bool = integrable?($\phi_E \times g$)

$\int_{E:\ \text{measurable\_set}} f$: (integrable?(E)): $\mathbb{R}$ =
$\int \phi_E \times f$

indefinite_emptyset: LEMMA $\int_{\emptyset} g = 0$

indefinite_fullset: LEMMA $\int_{\text{fullset}} f = \int f$

indefinite_eq_0: LEMMA
∀ (E: measurable_set, f: (integrable?(E))):
ae_in?(λ x: $f(x) > 0$)(E) $\wedge$ $\int \phi_E \times f = 0 \Rightarrow$
(mu_fin?(E) $\wedge$ $\mu(E) = 0$)

indefinite_eq: LEMMA
(∀ E: $\int_E f_1 = \int_E f_2$) $\Rightarrow$ $f_1 = f_2$ *a.e.*

indefinite_phi: LEMMA $\int_E \phi_F = \mu((E \cap F))$

indefinite_add: LEMMA
∀ (E: measurable_set, f_1, f_2: (integrable?(E))):
$\int_E f_1 + f_2 = \int_E f_1 + \int_E f_2$

indefinite_scal: LEMMA
$\forall\ (E\colon \text{measurable\_set},\ f\colon (\text{integrable?}(E)))\colon$
$\int_E (c \times f) \ = \ c \times \int_E f$

indefinite_opp: LEMMA
$\forall\ (E\colon \text{measurable\_set},\ f\colon (\text{integrable?}(E)))\colon$
$\int_E -f \ = \ -\int_E f$

indefinite_diff: LEMMA
$\forall\ (E\colon \text{measurable\_set},\ f_1,\ f_2\colon (\text{integrable?}(E)))\colon$
$\int_E f_1 - f_2 \ = \ \int_E f_1 - \int_E f_2$

indefinite_ae_eq: LEMMA
$f_1 = f_2\ \ a.e.\ \Leftrightarrow (\forall\ E\colon \int_E f_1 \ = \ \int_E f_2)$

indefinite_0_le: LEMMA $f \geq 0\ \ a.e.\ \Leftrightarrow (\forall\ E\colon 0 \ \leq \ \int_E f)$

indefinite_le: LEMMA
$f_1 \leq f_2\ \ a.e.\ \Leftrightarrow (\forall\ E\colon \int_E f_1 \ \leq \ \int_E f_2)$

indefinite_pm: LEMMA
$\forall\ (E\colon \text{measurable\_set},\ f\colon (\text{integrable?}(E)))\colon$
$\int_E f \ = \ \int_E f^+ - \int_E f^-$

indefinite_union: LEMMA
$\forall\ (E_1,\ E_2\colon \text{measurable\_set},\ f\colon (\text{integrable?}((E_1 \cup E_2))))\colon$
$\text{disjoint?}(E_1,\ E_2) \ \Rightarrow$
$\int_{(E_1 \cup E_2)} f \ = \ \int_{E_1} f + \int_{E_2} f$

indefinite_subset: LEMMA
$\forall\ (E_1,\ E_2\colon \text{measurable\_set},\ f\colon (\text{integrable?}(E_2)))\colon$
$(E_1 \subseteq E_2)\ \wedge\ f \geq 0\ \ a.e.\ \Rightarrow \int_{E_1} f \ \leq \ \int_{E_2} f$

indefinite_null: LEMMA $\int_N h \ = \ 0$

indefinite_countably_additive: LEMMA
$\text{series}(\lambda\ n:\ \int_{\text{DX}(n)} f) \longrightarrow \int_{\bigcup \text{DX}} f$

END indefinite_integral

40 measure_equality

measure_equality[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_def[T, S]

μ, ν: VAR measure_type

x: VAR T

f: VAR $[T \rightarrow \mathbb{R}]$

g: VAR $[T \rightarrow \mathbb{R}_{\geq 0}]$

E: VAR (S)

IMPORTING $\int$

measure_eq_isf?: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \Rightarrow$
(isf?[T, S, μ](f) $\Leftrightarrow$ isf?[T, S, ν](f))

measure_eq_isf: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \wedge$
(isf?[T, S, μ](f) $\vee$ isf?[T, S, ν](f))
$\Rightarrow$
(isf_integral[T, S, μ](f) $=$
isf_integral[T, S, ν](f))

measure_eq_nn_integrable?: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \Rightarrow$
(nn_integrable?[T, S, μ](g) $\Leftrightarrow$
nn_integrable?[T, S, ν](g))

measure_eq_nn_integral: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \wedge$
(nn_integrable?[T, S, μ](g) $\vee$ nn_integrable?[T, S, ν](g))
$\Rightarrow$
(nn_integral[T, S, μ](g) $=$ nn_integral[T, S, ν](g))

measure_eq_integrable?: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \Rightarrow$
(integrable?[T, S, μ](f) $\Leftrightarrow$ integrable?[T, S, ν](f))

measure_eq_integral: LEMMA
$(\forall\ E\colon\ \mu(E) = \nu(E)) \wedge$
(integrable?[T, S, μ](f) $\vee$ integrable?[T, S, ν](f))
$\Rightarrow (\int f\ =\ \int f)$

END measure_equality

41 measure_contraction

measure_contraction[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_def[T, S], measure_space[T, S]

$f \times g$: [T → ℝ]: MACRO [T → ℝ] = $f \times g$

μ: VAR measure_type

ν: VAR sigma_finite_measure

A, E: VAR measurable_set

f: VAR measurable_function

i: VAR ℕ

contraction(μ, A): measure_type = λ E: $\mu((A \cap E))$

fm_contraction(μ: measure_type, A: {E | $\mu(E)$`1}): finite_measure =
λ E: $\mu((A \cap E))$`2

sigma_finite_contraction_def: LEMMA
$\nu(E) = \sum \lambda$ i : (TRUE, fm_contraction(ν, A_of(ν)(i))(E))

IMPORTING isf, nn_integral, $\int$, indefinite_integral, integral_convergence

contraction_is_sigma_finite: JUDGEMENT contraction(ν, A) HAS_TYPE
sigma_finite_measure

contraction_isf: LEMMA
∀ (f: simple):
isf?[T, S, contraction(μ, A)](f) ⇔
isf?[T, S, μ](($\phi_A \times f$))

contraction_isf_integral: LEMMA
∀ (f: isf[T, S, contraction(μ, A)]):
isf_integral[T, S, contraction(μ, A)](f) =
isf_integral[T, S, μ](($\phi_A \times f$))

contraction_nn_integrable: LEMMA
∀ (f: nn_measurable):
nn_integrable?[T, S, contraction(μ, A)](f) ⇔
nn_integrable?[T, S, μ](($\phi_A \times f$))

contraction_nn_integral: LEMMA
∀ (f: nn_integrable[T, S, contraction(μ, A)]):
nn_integral[T, S, contraction(μ, A)](f) =
nn_integral[T, S, μ](($\phi_A \times f$))

contraction_integrable: LEMMA
  integrable?[T, S, contraction(μ, A)](f) $\Leftrightarrow$
   integrable?[T, S, μ](($\phi_A \times f$))

contraction_integral: LEMMA
  $\forall$ (f: integrable[T, S, contraction(μ, A)]):
    $\int f \ = \ \int (\phi_A \times f)$

END measure_contraction

42 measure_contraction_props

measure_contraction_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: measure_type]: THEORY
BEGIN

IMPORTING measure_props[T, S, μ], measure_contraction[T, S], integral_convergence_scaf

A: VAR disjoint_indexed_measurable

h: VAR measurable_function

x: VAR T

n: VAR $\mathbb{N}$

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

contraction_integrable_def: LEMMA
$\bigcup A$ = fullset[T] $\wedge$ ($\forall$ x: $h(x) \geq 0$) $\Rightarrow$
(integrable?[T, S, μ](h) $\Leftrightarrow$
(($\forall$ n: integrable?[T, S, contraction(μ, $A(n)$)](h)) $\wedge$
topological_convergence.convergent?
(series(λ n: $\int h$))))

END measure_contraction_props

43 sigma_finite_measure_props

sigma_finite_measure_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: sigma_finite_measure]: THEORY
BEGIN

IMPORTING measure_contraction_props[T, S, μ], measure_equality[T, S]

f: VAR nn_integrable[T, S, μ]

g: VAR integrable[T, S, μ]

h: VAR nn_measurable[T, S]

A: VAR (S)

x: VAR T

n: VAR $\mathbb{N}$

F: VAR sequence[[$T \to \mathbb{R}$]]

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

sfm_integrable: LEMMA
(($\forall$ n: integrable?[T, S, contraction(μ, A_of(μ)(n))](h)) $\wedge$
topological_convergence.convergent?(series(λ n: $\int h$)))
$\Leftrightarrow$ integrable?[T, S, μ](h)

sfm_integral: LEMMA series(λ n : $\int f$) $\longrightarrow \int f$

sfm_component_eq: LEMMA
to_measure(fm_contraction(μ, A_of(μ)(n)))(A) = contraction(μ, A_of(μ)(n))(A)

IMPORTING integral_convergence[T, S, μ]

sfm_monotone_convergence: LEMMA
increasing?(F) *a.e.* $\wedge$
($\forall$ n, x: $F(n)(x) \geq 0$) $\wedge$
($\forall$ n: integrable?[T, S, contraction(μ, P_of(μ)(n))]($F(n)$))
$\Rightarrow$
((($\exists$ g: $F \longrightarrow g$ *a.e.*) $\Leftrightarrow$ bounded?(λ n: $\int F(n)$)) $\wedge$
($\forall$ g: $F \longrightarrow g$ *a.e.* $\Rightarrow$ λ n : $\int F(n) \nearrow \int g$))

END sigma_finite_measure_props

Part VII
Product Measures

44 product_finite_measure

product_finite_measure[(IMPORTING subset_algebra_def) T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2]]: THEORY
BEGIN

IMPORTING product_sigma_def[T_1, T_2], product_sigma[T_1, T_2, S_1, S_2], measure_def[T_1, S_1],
measure_def[T_2, S_2], measure_def[[T_1, T_2], $S_1 \times S_2$], $\int$, finite_measure,
monotone_classes, integral_convergence

M: VAR $(S_1 \times S_2)$

x: VAR T_1

y: VAR T_2

X: VAR (S_1)

Y: VAR (S_2)

E: VAR sequence[$(S_1 \times S_2)$]

μ: VAR finite_measure[T_1, S_1]

ν: VAR finite_measure[T_2, S_2]

x_section_bounded: LEMMA
$0 \leq (\nu \circ \text{x\_section}(M))(x) \wedge$
$(\nu \circ \text{x\_section}(M))(x) \leq \nu(\text{fullset}[T_2])$

y_section_bounded: LEMMA
$0 \leq (\mu \circ \text{y\_section}(M))(y) \wedge$
$(\mu \circ \text{y\_section}(M))(y) \leq \mu(\text{fullset}[T_1])$

x_section_measurable: LEMMA
measurable_function?[T_1, S_1]($\nu \circ$ x_section(M))

y_section_measurable: LEMMA
measurable_function?[T_2, S_2]($\mu \circ$ y_section(M))

x_section_integrable: LEMMA
integrable?[T_1, S_1, to_measure(μ)]($\nu \circ$ x_section(M))

y_section_integrable: LEMMA
integrable?[T_2, S_2, to_measure(ν)]($\mu \circ$ y_section(M))

rectangle_measure1: LEMMA

$M \ = \ X \times Y \ \Rightarrow$
$\int \nu \circ \text{x\_section}(M) \ = \ \mu(X) \times \nu(Y)$

rectangle_measure2: LEMMA
$M \ = \ X \times Y \ \Rightarrow$
$\int \mu \circ \text{y\_section}(M) \ = \ \mu(X) \times \nu(Y)$

$\mu \times \nu$: finite_measure$[[T_1, \ T_2], \ S_1 \times S_2] \ =$
$\lambda \ M\colon \ \int \nu \circ \text{x\_section}(M)$

fm_times_alt: LEMMA
finite_measure?$[[T_1, \ T_2], \ S_1 \times S_2]$
$(\lambda \ M\colon \ \int \mu \circ \text{y\_section}(M))$

finite_product_alt: THEOREM
fm_times$(\mu, \ \nu)(M) \ = \ \int \mu \circ \text{y_section}(M)$

END product_finite_measure

45 product_measure

product_measure[(IMPORTING subset_algebra_def) T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2]]: THEORY

BEGIN

IMPORTING product_sigma[T_1, T_2, S_1, S_2], measure_contraction[T_1, S_1],
measure_contraction[T_2, S_2], measure_contraction[[T_1, T_2], $S_1 \times S_2$],
product_finite_measure[T_1, T_2, S_1, S_2], $\overline{\mathbb{R}}_{\geq 0}$@double_index[set[[$T_1$, T_2]]]

μ: VAR sigma_finite_measure[T_1, S_1]

ν: VAR sigma_finite_measure[T_2, S_2]

X: VAR (S_1)

Y: VAR (S_2)

M: VAR ($S_1 \times S_2$)

z: VAR [T_1, T_2]

i, j, n: VAR $\mathbb{N}$

product_measure_approx(μ, ν)(i, j): finite_measure[[T_1, T_2], $S_1 \times S_2$] =
fm_contraction[T_1, S_1](μ, A_of(μ)(i)) $\times$ fm_contraction [T_2, S_2](ν, A_of(ν)(j))

$\mu \times \nu$: sigma_finite_measure[[T_1, T_2], $S_1 \times S_2$] =
λ M:
$\sum \lambda$ i : $\sum \lambda$ j : to_measure(product_measure_approx(μ, ν)(i, j))(M)

m_times_alt: LEMMA
m_times(μ, ν)(M) = $\sum \lambda$ j : $\sum \lambda$ i : to_measure(product_measure_approx(μ, ν)(i, j))(M)

rectangle_measure: LEMMA
m_times(μ, ν)($X \times Y$) = $\mu(X) \times \nu(Y)$

phi1(X): simple[[T_1, T_2], $S_1 \times S_2$] =
$\phi_{X \times \text{fullset}[T_2]}$

phi2(Y): simple[[T_1, T_2], $S_1 \times S_2$] =
$\phi_{\text{fullset}[T_1] \times Y}$

END product_measure

Part VIII
Product Integrals

46 product_integral_def

product_integral_def[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2], μ: measure_type[T_1, S_1],
ν: measure_type[T_2, S_2]]: THEORY

BEGIN

IMPORTING $\int[T_1, S_1, \mu]$, $\int[T_2, S_2, \nu]$, reals@real_fun_ops[[T_1, T_2]]

h: VAR $[[T_1, T_2] \rightarrow \mathbb{R}]$

f: VAR integrable[T_1, S_1, μ]

g: VAR integrable[T_2, S_2, ν]

N_1: VAR null_set[T_1, S_1, μ]

N_2: VAR null_set[T_2, S_2, ν]

x: VAR T_1

y: VAR T_2

c: VAR $\mathbb{R}$

integrable1?(h): bool =
  $\exists\ N_1, f$:
    $\forall\ x$:
      $\neg\ (x \in N_1) \Rightarrow$
       integrable?($\lambda\ y$: $h(x, y)$) $\wedge$
        $\int \lambda\ y: h(x, y) = f(x)$

integrable2?(h): bool =
  $\exists\ N_2, g$:
    $\forall\ y$:
      $\neg\ (y \in N_2) \Rightarrow$
       integrable?($\lambda\ x$: $h(x, y)$) $\wedge$
        $\int \lambda\ x: h(x, y) = g(y)$

integrable1: TYPE+ = (integrable1?) CONTAINING ($\lambda\ x, y$: 0)

integrable2: TYPE+ = (integrable2?) CONTAINING ($\lambda\ x, y$: 0)

g_1, h_1: VAR integrable1

g_2, h_2: VAR integrable2

integrable1_zero: LEMMA integrable1?(λ x, y: 0)

integrable1_add: JUDGEMENT $+(g_1,\ h_1)$ HAS_TYPE integrable1

integrable1_scal: JUDGEMENT $\times(c,\ h_1)$ HAS_TYPE integrable1

integrable1_opp: JUDGEMENT $-(h_1)$ HAS_TYPE integrable1

integrable1_diff: JUDGEMENT $-(g_1,\ h_1)$ HAS_TYPE integrable1

integrable2_zero: LEMMA integrable2?(λ x, y: 0)

integrable2_add: JUDGEMENT $+(g_2,\ h_2)$ HAS_TYPE integrable2

integrable2_scal: JUDGEMENT $\times(c,\ h_2)$ HAS_TYPE integrable2

integrable2_opp: JUDGEMENT $-(h_2)$ HAS_TYPE integrable2

integrable2_diff: JUDGEMENT $-(g_2,\ h_2)$ HAS_TYPE integrable2

null_integrable1(h_1): [null_set[T_1, S_1, μ], integrable[T_1, S_1, μ]] =
choose({N_1, f |
$\forall$ x:
$\neg$ $(x \in N_1)$ $\Rightarrow$
integrable?(λ y: $h_1(x,\ y)$) $\wedge$
$\int \lambda$ y: $h_1(x,\ y) = f(x)$})

null_integrable2(h_2): [null_set[T_2, S_2, ν], integrable[T_2, S_2, ν]] =
choose({N_2, g |
$\forall$ y:
$\neg$ $(y \in N_2)$ $\Rightarrow$
integrable?(λ x: $h_2(x,\ y)$) $\wedge$
$\int \lambda$ x: $h_2(x,\ y) = g(y)$})

null_integral1_def: LEMMA
$(N_1,\ f)$ = null_integrable1(h_1) $\wedge$ $(\neg\ (x \in N_1))$ $\Rightarrow$
(integrable?(λ y: $h_1(x,\ y)$) $\wedge$
$\int \lambda$ y: $h_1(x,\ y) = f(x)$)

null_integral2_def: LEMMA
$(N_2,\ g)$ = null_integrable2(h_2) $\wedge$ $(\neg\ (y \in N_2))$ $\Rightarrow$
(integrable?(λ x: $h_2(x,\ y)$) $\wedge$
$\int \lambda$ x: $h_2(x,\ y) = g(y)$)

integral1(h_1): integrable[T_1, S_1, μ] =
λ x:
LET $(N_1,\ f)$ = null_integrable1(h_1) IN
IF $(x \in N_1)$ THEN 0 ELSE $f(x)$ ENDIF

integral2(h_2): integrable[T_2, S_2, ν] =
λ y:

LET $(N_2, \ g) = $ null_integrable2(h_2) IN
IF $(y \in N_2)$ THEN 0 ELSE $g(y)$ ENDIF

integral1_zero: LEMMA integral1$(\lambda \ x, \ y: \ 0) = (\lambda \ x: \ 0)$

integral1_add: LEMMA
integral1$(g_1 + h_1) =$ integral1$(g_1) +$ integral1(h_1) *a.e.*

integral1_scal: LEMMA
integral1$(c \times h_1) = c \times$ integral1(h_1) *a.e.*

integral1_opp: LEMMA
integral1$(-(h_1)) = -$integral1(h_1) *a.e.*

integral1_diff: LEMMA
integral1$(g_1 - h_1) =$ integral1$(g_1) -$ integral1(h_1) *a.e.*

integral2_zero: LEMMA integral2$(\lambda \ x, \ y: \ 0) = (\lambda \ y: \ 0)$

integral2_add: LEMMA
integral2$(g_2 + h_2) =$ integral2$(g_2) +$ integral2(h_2) *a.e.*

integral2_scal: LEMMA
integral2$(c \times h_2) = c \times$ integral2(h_2) *a.e.*

integral2_opp: LEMMA
integral2$(-(h_2)) = -$integral2(h_2) *a.e.*

integral2_diff: LEMMA
integral2$(g_2 - h_2) =$ integral2$(g_2) -$ integral2(h_2) *a.e.*

END product_integral_def

47 finite_fubini_scaf

finite_fubini_scaf[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2], μ: finite_measure[T_1, S_1],
ν: finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING product_sigma[T_1, T_2, S_1, S_2], measure_def[T_1, S_1], measure_def[T_2, S_2],
measure_def[[T_1, T_2], $S_1 \times S_2$], product_finite_measure[T_1, T_2, S_1, S_2]

IMPORTING nn_integral[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

g: VAR nn_integrable

i: VAR isf

n: VAR nn_isf

E: VAR ($S_1 \times S_2$)

IMPORTING $\int$[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

f: VAR integrable

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

m: VAR measurable_function[[T_1, T_2], $S_1 \times S_2$]

x: VAR T_1

y: VAR T_2

IMPORTING $\int$[T_1, S_1, to_measure(μ)], $\int$[T_2, S_2, to_measure(ν)]

measurable_x_section: LEMMA
measurable_function?[T_2, S_2](λ y: $m(x, y)$)

measurable_y_section: LEMMA
measurable_function?[T_1, S_1](λ x: $m(x, y)$)

isf_x_section: LEMMA isf?(λ y: $i(x, y)$)

isf_y_section: LEMMA isf?(λ x: $i(x, y)$)

integral_phi1: LEMMA
(λ x: isf_integral[T_2, S_2, to_measure(ν)](λ y: $\phi(E)(x, y)$)) =
$\nu \circ$ x_section(E)

integral_phi2: LEMMA
(λ y: isf_integral[T_1, S_1, to_measure(μ)](λ x: $\phi(E)(x, y)$)) =
$\mu \circ$ y_section(E)

integral_phi3: LEMMA
isf_integral(ϕ_E) =
$\int \lambda\ x:$ isf_integral($\lambda\ y:\ \phi(E)(x,\ y)$)

integral_phi4: LEMMA
isf_integral(ϕ_E) =
$\int \lambda\ y:$ isf_integral($\lambda\ x:\ \phi(E)(x,\ y)$)

isf_integral_x: LEMMA
integrable?($\lambda\ x$: isf_integral($\lambda\ y$: $i(x,\ y)$))

isf_integral_y: LEMMA
integrable?($\lambda\ y$: isf_integral($\lambda\ x$: $i(x,\ y)$))

isf_fubini_tonelli_3: LEMMA
isf_integral(i) =
$\int \lambda\ x:$ isf_integral($\lambda\ y:\ i(x,\ y)$)

isf_fubini_tonelli_4: LEMMA
isf_integral(i) =
$\int \lambda\ y:$ isf_integral($\lambda\ x:\ i(x,\ y)$)

END finite_fubini_scaf

48 finite_fubini_tonelli

finite_fubini_tonelli[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2],
μ: finite_measure[T_1, S_1], ν: finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING finite_fubini_scaf[T_1, T_2, S_1, S_2, μ, ν],
product_integral_def[T_1, T_2, S_1, S_2, to_measure(μ), to_measure(ν)],
integral_convergence, indefinite_integral

g: VAR
nn_integrable
[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

finite_fubini_tonelli_1: LEMMA integrable?(h) $\Leftrightarrow$ integrable1?(h)

finite_fubini_tonelli_2: LEMMA integrable?(h) $\Leftrightarrow$ integrable2?(h)

finite_fubini_tonelli_3: LEMMA $\int g = \int$ integral1(g)

finite_fubini_tonelli_4: LEMMA $\int g = \int$ integral2(g)

END finite_fubini_tonelli

49 finite_fubini

finite_fubini[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2], μ: finite_measure[T_1, S_1],
ν: finite_measure[T_2, S_2]]: THEORY
BEGIN

IMPORTING sigma_algebra[T_1, S_1], sigma_algebra[T_2, S_2],
finite_fubini_tonelli[T_1, T_2, S_1, S_2, μ, ν],
finite_integral[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$],
finite_integral[T_1, S_1, μ], finite_integral[T_2, S_2, ν]

f: VAR
integrable
[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

finite_integrable_is_integrable1: LEMMA integrable1?(f)

finite_integrable_is_integrable2: LEMMA integrable2?(f)

finite_fubini1: COROLLARY $\int f = \int$ integral1(f)

finite_fubini2: COROLLARY $\int f = \int$ integral2(f)

h: VAR bounded_measurable[[T_1, T_2], $S_1 \times S_2$]

x: VAR T_1

y: VAR T_2

integrable_x_section: LEMMA integrable?(λ y: $h(x, y)$)

integrable_y_section: LEMMA integrable?(λ x: $h(x, y)$)

integrable_integral_x_section: LEMMA
integrable?(λ x: $\int \lambda\ y:\ h(x, y)$)

integrable_integral_y_section: LEMMA
integrable?(λ y: $\int \lambda\ x:\ h(x, y)$)

integral_integral_x_section: LEMMA
$\int \lambda\ x:\ \int \lambda\ y:\ h(x, y) = \int h$

integral_integral_y_section: LEMMA
$\int \lambda\ y:\ \int \lambda\ x:\ h(x, y) = \int h$

END finite_fubini

50 fubini_tonelli_scaf

fubini_tonelli_scaf[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2],
μ: sigma_finite_measure[T_1, S_1],
ν: sigma_finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING product_measure[T_1, T_2, S_1, S_2], $\int$[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$]

E: VAR ($S_1 \times S_2$)

X: VAR (S_1)

Y: VAR (S_2)

x: VAR T_1

y: VAR T_2

i, j, n: VAR $\mathbb{N}$

IMPORTING product_integral_def[T_1, T_2, S_1, S_2, μ, ν], measure_contraction_props,
measure_equality, finite_fubini_tonelli, finite_fubini, indefinite_integral,
$\overline{\mathbb{R}}_{\geq 0}$@double_nn_sequence, sigma_finite_measure_props

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

IMPORTING product_integral_def, sigma_finite_measure_props

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

product_measure_contraction: LEMMA
contraction($\mu \times \nu$, $X \times Y$)(E) = m_times(contraction(μ, X), contraction(ν, Y))(E)

product_sfm_contraction: LEMMA
contraction($\mu \times \nu$, A_of(μ)(i) $\times$ A_of(ν)(j))(E) = product_measure_approx(μ, ν)(i, j)(E)

product_measure_contraction_n: LEMMA
m_times(contraction(μ, P_of(μ)(n)), contraction(ν, P_of(ν)(n)))(E) = to_measure(fm_contraction(μ, P_of(μ)(n))

fubini_tonelli_scaf1: LEMMA
(integrable?(h) $\Leftrightarrow$ integrable1?(h)) $\wedge$
(integrable?(h) $\Rightarrow$
$\int h$ = $\int$ integral1[T_1, T_2, S_1, S_2, μ, ν](h))

fubini_tonelli_scaf2: LEMMA
(integrable?(h) $\Leftrightarrow$ integrable2?(h)) $\wedge$
(integrable?(h) $\Rightarrow$
$\int h$ = $\int$ integral2[T_1, T_2, S_1, S_2, μ, ν](h))

END fubini_tonelli_scaf

51 fubini_tonelli

fubini_tonelli$\big[$(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra$[T_1]$, S_2: sigma_algebra$[T_2]$,
μ: sigma_finite_measure$[T_1, S_1]$, ν: sigma_finite_measure$[T_2, S_2]\big]$: THEORY

BEGIN

IMPORTING product_measure$[T_1, T_2, S_1, S_2]$, $\int\big[[T_1, T_2], S_1 \times S_2, \mu \times \nu\big]$

g: VAR nn_integrable

h: VAR nn_measurable$\big[[T_1, T_2], S_1 \times S_2\big]$

x: VAR T_1

y: VAR T_2

IMPORTING product_integral_def$[T_1, T_2, S_1, S_2, \mu, \nu]$,
fubini_tonelli_scaf$[T_1, T_2, S_1, S_2, \mu, \nu]$

fubini_tonelli_1: THEOREM integrable?(h) $\Leftrightarrow$ integrable1?(h)

fubini_tonelli_2: THEOREM integrable?(h) $\Leftrightarrow$ integrable2?(h)

fubini_tonelli_3: THEOREM
$\int g = \int$ integral1$[T_1, T_2, S_1, S_2, \mu, \nu](g)$

fubini_tonelli_4: THEOREM
$\int g = \int$ integral2$[T_1, T_2, S_1, S_2, \mu, \nu](g)$

END fubini_tonelli

52 fubini

fubini[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2], μ: sigma_finite_measure[T_1, S_1],
ν: sigma_finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING fubini_tonelli[T_1, T_2, S_1, S_2, μ, ν]

f: VAR integrable[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$]

x: VAR T_1

y: VAR T_2

integrable_is_integrable1: LEMMA integrable1?(f)

integrable_is_integrable2: LEMMA integrable2?(f)

fubini1: LEMMA
$\int f = \int$ integral1[T_1, T_2, S_1, S_2, μ, ν](f)

fubini2: LEMMA
$\int f = \int$ integral2[T_1, T_2, S_1, S_2, μ, ν](f)

END fubini